The path of a basketball in flight can be described by a non-linear relationship. What two variables could you relate in a rule for this relationship?
1 Factorise each quadratic expression.
   a \( x^2 + 7x \)
   b \( x^2 - 9 \)
   c \( x^2 + 5x + 6 \)
   d \( -x^2 - 3x \)

2 Substitute these values into each expression.
   i \( x = 2 \)
   ii \( x = -3 \)
   iii \( x = 0 \)
   a \( x^2 - 4x \)
   b \( x^2 - 4 \)
   c \( x^2 - 2x + 1 \)
   d \( (x - 4)(x + 3) \)

3 Simplify each expression.
   a \( 3x^2 + 4x + x^2 + 1 \)
   b \( x^2 - 7x + 2x - 5 \)
   c \( x - x^2 + 2 - 6x - 8 \)
   d \( 2x^2 - 2 + 2x + 2 - 2x \)

4 a Copy and complete this table of values for each rule listed below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 2x )</td>
<td>( 3 )</td>
<td>( 2 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

   i \( y = 2x \)
   ii \( y = x^2 \)
   b Plot the graph of each rule.
   c Identify whether each graph shows a linear relationship.

5 Write the rule for each of the linear graphs shown.

6 Plot the point A(2, 3) on a Cartesian plane. Find the coordinates of a new point after each transformation has been performed on the point A.
   a reflect in the \( x \)-axis
   b translate 4 units right
   c translate 5 units down
   d translate 1 unit up
   e translate 6 units left
   f translate 2 units left and 3 units up

7 Find the \( x \) - and \( y \) -intercepts of the graphs of each relationship.
   a \( y = x - 4 \)
   b \( y = 2x + 6 \)

8 Find the gradient of the graph drawn in question 4ai for \( y = 2x \).
4A Solving quadratic equations

Start thinking!

1. What is a quadratic expression? Provide three examples.

2. You can also consider quadratic equations.
   a. What is the difference between a quadratic equation and a quadratic expression?
   b. Which of these are quadratic equations?
      i. $x^2 - 4 = 0$
      ii. $x - 4 = 0$
      iii. $x^2 + 7x = 0$
      iv. $x^2 + 3x - 4$
   c. Explain the difference between a quadratic equation and a linear equation.

3. One method to solve a quadratic equation is to use a guess, check and improve strategy (trial and error). With a partner, solve each equation. (Hint: each equation has two solutions.)
   a. $x^2 - 4 = 0$
   b. $(x - 3)(x - 5) = 0$
   c. $x^2 + 7x = 0$
   d. $x^2 + 3x - 4 = 0$

4. Another method involves the Null Factor Law.
   a. Calculate each of these.
      i. $5 \times 0$
      ii. $0 \times -4$
      iii. $0 \times 0$
      iv. $x \times 0$
      v. $0 \times (x - 7)$
      vi. $(x + 2) \times 0$
   b. Use your answers to part a to help you work out the value of $x$ in each equation.
      i. $2 \times x = 0$
      ii. $x \times 6 = 0$
      iii. $3 \times (x - 1) = 0$
      iv. $(x + 5) \times 8 = 0$
   c. Copy and complete this sentence: The Null Factor Law states that if the product of two factors equals _______, then one or both of the factors must equal _______.

5. Consider the Null Factor Law with $(x - 3)(x - 5) = 0$.
   a. What are the two factors that form the product on the left side of the equation?
   b. Since the right side of the equation equals 0, what do you know about $(x - 3)$ and $(x - 5)$?
   c. Copy and complete the workings on the right to solve $(x - 3)(x - 5) = 0$ using the Null Factor Law.
   d. Check your two solutions. Explain how you did this.

   $(x - 3)(x - 5) = 0$
   $x - 3 = 0$ or $x - 5 = ___$
   $x = 3$ or $x = ___$

KEY IDEAS

- The general form of a quadratic equation is $ax^2 + bx + c = 0$ or $(x + m)(x + n) = 0$.
- The Null Factor Law states that if the product of two factors is 0, then one or both factors are 0. For example, if $a \times b = 0$ then $a = 0$ or $b = 0$ or both $a$ and $b$ are 0.
- A quadratic equation can be solved using the Null Factor Law if one side of the equation is in factor form and the other equals 0. Two linear equations are produced which are easy to solve. For example, $(x + m)(x + n) = 0$ has two solutions: $x = -m$ or $x = -n$.
- To obtain the product of two factors on one side of the equation, the quadratic expression needs to be factorised.
EXERCISE 4A  Solving quadratic equations

1 Which of these are quadratic equations?
   a $x^2 - 2 = 0$
   b $3(x + 1) = 0$
   c $x^2 + x = 5$
   d $x^2 + 5x + 6$
   e $2x^2 + x - 4 = 0$
   f $x^3 + 8 = 0$
   g $6x + 1 = 2x - 5$
   h $x^2 + 7x = x - 3$

EXAMPLE 4A-1  Solving quadratic equations using the Null Factor Law

Solve each equation.
   a $(x + 6)(x - 2) = 0$
   b $x(x - 4) = 0$

THINK
   a 1 Write the equation. Check that the left side (LS) is in factor form (yes) and the right side (RS) equals 0 (yes).
   2 Apply the Null Factor Law.
   3 Solve each linear equation.

WRITE
   a $(x + 6)(x - 2) = 0$
      $x + 6 = 0$ or $x - 2 = 0$
      $x = -6$ or $x = 2$
   b $x(x - 4) = 0$
      $x = 0$ or $x - 4 = 0$
      $x = 0$ or $x = 4$

2 Copy and complete the steps shown to solve each quadratic equation using the Null Factor Law.

   a $(x + 7)(x - 4) = 0$
      $x + 7 = 0$ or $x - 4 = __$
      $x = __$ or $x = __$
   b $(x - 2) = 0$
      $x = __$ or $x - 2 = __$
      $x = __$ or $x = __$
   c $(x + 5)(x - 5) = 0$
      $(x + 5) = __$ or $(x - 5) = __$
      $x = __$ or $x = __$

3 Solve each equation. Show all steps of working.
   a $(x + 2)(x - 3) = 0$
   b $(x - 7)(x - 1) = 0$
   c $(x + 4)(x - 4) = 0$
   d $x(x - 6) = 0$
   e $(x + 5)(x + 1) = 0$
   f $x(x + 2) = 0$
   g $(x - 8)(x + 8) = 0$
   h $(x + 1)(x - 7) = 0$
   i $x(x - 11) = 0$
   j $(x + 3)(x - 5) = 0$
   k $(x - 2)(x - 2) = 0$
   l $(x + 5)(x + 5) = 0$
Solve each equation.

\[
\begin{align*}
\text{a} & \quad x^2 - 9 = 0 \\
\text{b} & \quad x^2 - 3x - 10 = 0 \\
\text{c} & \quad x^2 + 5x = 0 \\
\text{d} & \quad x^2 - 3x = 0 \\
\text{e} & \quad x^2 - 36 = 0 \\
\text{f} & \quad x^2 + 10x + 21 = 0 \\
\text{g} & \quad x^2 - 2x - 8 = 0 \\
\text{h} & \quad x^2 - 1 = 0 \\
\text{i} & \quad x^2 + 8x = 0 \\
\text{j} & \quad x^2 - 4x + 3 = 0 \\
\text{k} & \quad x^2 + 6x + 9 = 0 \\
\text{l} & \quad x^2 - 2x + 1 = 0
\end{align*}
\]

**THINK**

\[
\begin{align*}
\text{a} & \quad 1 \quad \text{Write the equation. Check that the LS is in factor form (no) and the right side equals 0 (yes).} \\
\text{b} & \quad 1 \quad \text{Write the equation. Check that the LS is in factor form (no) and the RS equals 0 (yes).}
\end{align*}
\]

**WRITE**

\[
\begin{align*}
\text{a} & \quad x^2 - 9 = 0 \\
\text{b} & \quad x^2 - 3x - 10 = 0 \\
\text{c} & \quad (x + 3)(x - 3) = 0 \\
\text{d} & \quad (x + 2)(x - 5) = 0 \\
\text{e} & \quad x + 3 = 0 \text{ or } x - 3 = 0 \\
\text{f} & \quad x + 2 = 0 \text{ or } x - 5 = 0 \\
\text{g} & \quad x = -3 \text{ or } x = 3 \\
\text{h} & \quad x = -2 \text{ or } x = 5
\end{align*}
\]

**5** Use substitution to check that the solutions found in question 4 are correct.

**6** For each quadratic equation:

\[
\begin{align*}
\text{i} & \quad \text{solve the equation} \\
\text{ii} & \quad \text{compare your solution to that obtained using the guess, check and improve strategy in question 3 of 4A Start thinking! on page 154.}
\end{align*}
\]

\[
\begin{align*}
\text{a} & \quad x^2 - 4 = 0 \\
\text{b} & \quad (x - 3)(x - 5) = 0 \\
\text{c} & \quad x^2 + 7x = 0 \\
\text{d} & \quad x^2 + 3x - 4 = 0
\end{align*}
\]

**7** Decide whether the value given in square brackets is a solution to the equation.

\[
\begin{align*}
\text{a} & \quad (x - 4)(x - 5) = 0 \quad [x = 5] \\
\text{b} & \quad (x + 2)(x - 8) = 0 \quad [x = 2] \\
\text{c} & \quad x(x - 6) = 0 \quad [x = 3] \\
\text{d} & \quad x^2 + 8x + 7 = 0 \quad [x = -1] \\
\text{e} & \quad x^2 - 4x + 4 = 0 \quad [x = -2] \\
\text{f} & \quad x^2 - 49 = 0 \quad [x = 7] \\
\text{g} & \quad x^2 - 2x - 15 = 0 \quad [x = -3] \\
\text{h} & \quad x^2 - 8x + 12 = 0 \quad [x = -4]
\end{align*}
\]

**8**

\[
\begin{align*}
\text{a} & \quad \text{Why is } -x(x - 3) = 0 \text{ equivalent to } x(x - 3) = 0? \\
\text{b} & \quad \text{Solve } x(x - 3) = 0 \text{ and hence write the solutions for } -x(x - 3) = 0. \\
\text{c} & \quad \text{Use substitution to check that you have the correct solutions.}
\end{align*}
\]
9 a Why is $-2(x + 4)(x - 5) = 0$ equivalent to $(x + 4)(x - 5) = 0$?
    b Solve $(x + 4)(x - 5) = 0$ and hence write the solutions for $-2(x + 4)(x - 5) = 0$.
    c Use substitution to check that you have the correct solutions.

**EXAMPLE 4A-3**

Solving quadratic equations after first dividing both sides by a negative number

Solve each quadratic equation.

<table>
<thead>
<tr>
<th>a</th>
<th>$-4(x - 3)(x - 5) = 0$</th>
<th>b</th>
<th>$-x^2 - 11x - 18 = 0$</th>
</tr>
</thead>
</table>

**THINK**

a 1 Divide both sides of the equation by $-4$.

b 1 Take out $-1$ as a common factor on the LS.

2 Apply the Null Factor Law and solve each linear equation.

2 Divide both sides of the equation by $-1$.

3 Factorise the LS of the equation.

4 Apply the Null Factor Law and solve each linear equation.

**WRITE**

a $-4(x - 3)(x - 5) = 0$
   $$(x - 3)(x - 5) = 0$$
   $x - 3 = 0$ or $x - 5 = 0$
   $x = 3$ or $x = 5$

b $-x^2 - 11x - 18 = 0$
   $$-x^2 + 11x + 18 = 0$$
   $(x + 9)(x + 2) = 0$
   $x + 9 = 0$ or $x + 2 = 0$
   $x = -9$ or $x = -2$

10 Solve each quadratic equation.

a $-x(x + 9) = 0$
   b $-(x + 8)(x - 2) = 0$
   c $-3(x - 1)(x - 4) = 0$
   d $-7(x + 6)(x - 6) = 0$
   e $-x^2 - 10x - 21 = 0$
   f $-5x^2 - 5x + 10 = 0$

11 a How many solutions does $(x - 4)(x - 7) = 0$ have?
   b How is this different from the number of solutions a linear equation has?
   c Does every quadratic equation have two solutions? Discuss this with a classmate.
   d How many solutions does $(x - 4)(x - 4) = 0$ have? Explain.
   e How many solutions does $x^2 + 4 = 0$ have? Explain.
   f Summarise your findings. How many solutions can a quadratic equation have?

12 Find the solution/s to each equation.

a $x^2 - 4 = 0$
   b $x^2 + 3x - 10 = 0$
   c $x^2 - 6x + 9 = 0$
   d $x^2 + 1 = 0$
   e $x^2 + 7x = 0$
   f $x^2 + 12x + 32 = 0$
   g $x^2 - x - 72 = 0$
   h $x^2 + 2x + 5 = 0$

**NOTE** Some of these equations cannot be solved.
13 Solve each quadratic equation. (Hint: first rearrange each equation into the form \(ax^2 + bx + c = 0\).)

a \(x^2 + 2x = 3\)  
b \(x^2 - 20 = x\)  
c \(x^2 = 25\)  
d \(x^2 + 4x + 11 = 7\)  
e \(x^2 + 10x = 2x\)  
f \(x^2 - 4x = 36 - 4x\)  
g \(4x - x^2 = x\)  
h \(x^2 + 41 = 12x + 5\)  
i \(x^2 - x + 21 = -11x\)  
j \(x(x - 7) = 8\)  
k \((x - 3)^2 = 1\)  
l \(4x + 12 = x^2\)

14 Use the ‘solve’ function of a calculator or other digital technology to solve each quadratic equation in question 13 and compare your solutions.

15 Louise borrowed some money from her brother Rick. This can be represented by the quadratic relationship \(y = (x - 10)^2\), where \(y\) is the amount owed in dollars after \(x\) weeks.

a How much money did Louise borrow from Rick? (Hint: find \(y\) when \(x = 0\).)

b How long did it take for Louise to repay the loan? (Hint: find \(x\) when \(y = 0\).)

c Did Louise repay the same amount each week? Explain.

16 Alec throws a tennis ball back on to the court from the spectator stand. The height of the ball above the surface of the tennis court can be represented by the quadratic relationship \(h = -(t + 2)(t - 4)\), where \(h\) is the height in metres after \(t\) seconds in the air.

a What is the height of the ball after:
   i 1 second?  
   ii 2 seconds?

b What is the height of the ball when Alec releases it from his hand?

c How long does it take for the ball to hit the tennis court after it is thrown? (Hint: what is the height of the ball when it hits the tennis court?)

d Explain why there is only one time value for your answer to part c even though you have solved a quadratic equation that has two solutions.

17 Frank keeps track of the number of goals he scores in each match of his club’s football season. It follows the quadratic relationship \(g = n^2 - 8n + 17\), where \(n\) is the number of the match from the start of the season and \(g\) is the number of goals he scored in that match.

a How many goals did Frank score in the first match of the season? (Hint: find \(g\) when \(n = 1\).)

b How many goals did he score in the fifth match of the season?

c In which matches did he score exactly five goals? (Hint: find \(n\) when \(g = 5\).)

d In which match did he score one goal only?
18 This rectangular mouse pad has the dimensions shown. Its length is 8 cm longer than its width.
   a Write an expression for the area of the mouse pad. (Hint: area of a rectangle is length $\times$ width.)
   b Write the expression in expanded form without brackets.
   c The area is estimated to be 560 cm$^2$.
      Write an equation for the area of the mouse pad.
   d Show how the equation can be written as $(x + 28)(x - 20) = 0$. (Hint: write the equation in the general form of $ax^2 + bx + c = 0$ before factorising.)
   e Solve the equation. Which value of $x$ is a feasible solution in this scenario? Explain.
   f Write the dimensions of the mouse pad.

19 The area of a rectangular sand pit is 35 m$^2$. The length is 2 m longer than the width.
   a Write a quadratic equation to represent this scenario. (Hint: let $x$ represent the width of the sand pit in metres.)
   b Solve the quadratic equation.
   c Write the dimensions of the sand pit.

20 The width of a laptop screen is 12 cm less than its length. Write an equation to represent the scenario where the area of the screen is 640 cm$^2$ and then solve it to find the dimensions of the screen.

21 Write a quadratic equation to match each set of solutions.
   a $x = 1$ and $x = 2$
   b $x = 0$ and $x = 10$
   c $x = -5$ and $x = 3$

22 Write another two quadratic equations for each set of solutions in question 21.

23 Solve each equation.
   a $-x^2 + 6x = 0$
   b $x - x^2 = 0$
   c $-x^2 - 2x + 3 = 0$
   d $-x^2 + 5x - 6 = 0$

Reflect
Why is the Null Factor Law useful when solving quadratic equations?
4B Plotting quadratic relationships

Start thinking!

1 Copy and complete this table of values using the rule for the quadratic relationship \( y = x^2 + 4x - 5 \). The first few \( y \) values have been calculated for you.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>7</td>
<td>0</td>
<td>-5</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

2 Plot the values for this relationship on a Cartesian plane. The first few points have been plotted for you.

3 Join the points you have plotted with a smooth line and describe the trend you see.

4 Does the graph show a linear relationship or a non-linear relationship between the variables? Explain.

The graph of a quadratic relationship is called a **parabola**.

5 A parabola changes direction at its **turning point**. If this is at its lowest point, the parabola has a **minimum turning point**. If this is at its highest point, the parabola has a **maximum turning point**.
   a What type of turning point does your graph have?
   b What are the coordinates of the turning point?

6 a Rule a vertical line on your graph through the turning point. Why do you think this is called the **axis of symmetry**?
   b What is the equation of the axis of symmetry? (Hint: the equation or rule will be of the form \( x = \ldots \))

7 a How many \( y \)-intercepts does your parabola have? List the coordinates of the \( y \)-intercept/s.
   b How many \( x \)-intercepts does your parabola have? List the coordinates of the \( x \)-intercept/s.

**KEY IDEAS**

- The graph of a quadratic relationship is a parabola.
- Creating a table of values helps you work out the coordinates of points to be plotted.
- These features can be identified from the graph:
  - type (or nature) of the turning point
  - coordinates of the turning point
  - equation of the axis of symmetry
  - \( x \)- and \( y \)-intercepts.
EXERCISE 4B  Plotting quadratic relationships

EXAMPLE 4B-1  Plotting a parabola

Plot the graph of \( y = -x^2 + 2x + 3 \) after completing a table for \( x \) values from \(-2\) to \(4\).

**THINK**

1. Create a table of values. Substitute each \( x \) value into the rule to find the corresponding \( y \) value.
2. Use grid paper to draw a Cartesian plane. Plot the points and join them with a smooth line. Label the graph with its rule.

**WRITE**

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-5)</td>
<td>(0)</td>
<td>(3)</td>
<td>(4)</td>
<td>(3)</td>
<td>(0)</td>
<td>(-5)</td>
</tr>
</tbody>
</table>

1. For each quadratic relationship:
   i. copy and complete the table of values
   ii. plot the graph.
   a. \( y = x^2 + 2x - 8 \)
   b. \( y = 9 - x^2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(3)</td>
<td>(4)</td>
<td>(3)</td>
<td>(0)</td>
<td>(-5)</td>
<td>(0)</td>
<td>(3)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(8)</td>
<td>(1)</td>
<td>(-2)</td>
<td>(-3)</td>
<td>(-4)</td>
<td>(-5)</td>
<td>(-6)</td>
<td>(-7)</td>
<td>(-8)</td>
</tr>
</tbody>
</table>

2. Plot the graph of each quadratic relationship after completing a table of values. You may like to use the suggested \( x \) values in your table.
   a. \( y = -x^2 - 6x - 5 \)  
      (\( x \) values from \(-6\) to \(0\))
   b. \( y = x^2 - 4 \)  
      (\( x \) values from \(-3\) to \(3\))
   c. \( y = x^2 - 2x - 15 \)  
      (\( x \) values from \(-4\) to \(6\))
   d. \( y = x^2 + 4x \)  
      (\( x \) values from \(-5\) to \(1\))
   e. \( y = -x^2 + 2x \)  
      (\( x \) values from \(-1\) to \(3\))
   f. \( y = x^2 - 6x + 9 \)  
      (\( x \) values from \(0\) to \(6\))
EXAMPLE 4B-2 Identifying features of a parabola

For the graph shown, identify:

a whether the parabola has a minimum or maximum turning point
b the coordinates of the turning point
c the equation of the axis of symmetry
d the y-intercept
e the x-intercepts.

THINK

a Locate the point where the graph changes direction. It is at the highest point on the parabola, so it is a maximum turning point.
b Write the x- and y-coordinates of the turning point.
c Locate the axis of symmetry. This is the vertical line that ‘cuts’ the parabola exactly in half.
d Locate the y-intercept. This is where the parabola crosses the y-axis.
e Locate the x-intercepts. This is where the parabola crosses the x-axis.

WRITE

a Parabola has a maximum turning point.
b Coordinates of turning point are (1, 4).
c The equation of the axis of symmetry is \( x = 1 \).
d y-intercept is 3.
The coordinates of the y-intercept are (0, 3).
e x-intercepts are −1 and 3.
The coordinates of the x-intercepts are (−1, 0) and (3, 0).

For the graph shown, identify:

a whether the parabola has a minimum or maximum turning point
b the coordinates of the turning point
c the equation of the axis of symmetry
d the y-intercept
e the x-intercepts.

For each graph drawn in question 1, identify:

i whether the parabola has a minimum or maximum turning point
ii the coordinates of the turning point
iii the equation of the axis of symmetry
iv the y-intercept
v the x-intercepts.
5 For each graph drawn in question 2, identify:
   i whether the parabola has a minimum or maximum turning point
   ii the coordinates of the turning point
   iii the equation of the axis of symmetry
   iv the y-intercept
   v the x-intercepts.

6 Produce each graph in questions 1 and 2 using digital technology.

7 Match each graph with its rule from the list below.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( y = x^2 + 4x - 5 )</td>
</tr>
<tr>
<td>b</td>
<td>( y = x^2 - 4x )</td>
</tr>
<tr>
<td>c</td>
<td>( y = -x^2 - 4x + 5 )</td>
</tr>
<tr>
<td>d</td>
<td>( y = -x^2 + 4 )</td>
</tr>
<tr>
<td>e</td>
<td>( y = x^2 - 4x - 5 )</td>
</tr>
<tr>
<td>f</td>
<td>( y = x^2 - 4x - 5 )</td>
</tr>
</tbody>
</table>

8 A parabola with a minimum turning point is described as upright. Its shape is similar to the shape of \( y = x^2 \). A parabola with a maximum turning point is described as inverted. Its shape is upside down compared with that of \( y = x^2 \). Identify each parabola in question 7 as upright or inverted.
9 Plot the graph of each quadratic relationship after completing the table of values shown. Hence, identify:

i whether the parabola is upright or inverted

ii the type of turning point and its coordinates

iii the y-intercept

iv the x-intercepts.

a \( y = 3x^2 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d \( y = 2x^2 - 4x - 6 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-1</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

e \( y = -\frac{1}{2}x^2 + 4x - 6 \)

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

e \( y = 3x^2 + 12x + 12 \)

<table>
<thead>
<tr>
<th>x</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

10 Lily’s cap fell to the ground while she was on a roller coaster ride. The position of the cap during the time it was falling can be described by the relationship \( h = 100 - 4t^2 \) where \( h \) was the height of the cap above the ground in metres after \( t \) seconds.

a Plot the graph of the relationship for \( t \) values from 0 to 6.

b Why didn’t you draw the full parabola for this scenario?

c What was the height of the cap after:

i 2 s?  ii 3 s?

d At what height off the ground did the cap start to fall?

e How long did it take for the cap to hit the ground?

11 Kim throws a javelin. The position of the tip of the javelin can be represented by the quadratic relationship

\( y = -0.05x^2 + 1.5x + 1.55 \), where \( y \) is the height above the ground and \( x \) is the horizontal distance from where the javelin was thrown. Both \( x \) and \( y \) are in metres.

a Plot the graph of this relationship. Use 0, 5, 10, 15, …, 40 as the \( x \) values in the table.

b What was the greatest height reached by the javelin?

c At what height off the ground was the javelin thrown?

d What horizontal distance did the javelin travel before hitting the ground?
12. Look at the graphs of these three quadratic relationships.

\[ y = x^2 - x - 2 \quad \text{ii} \quad y = -x^2 + 4x - 4 \quad \text{iii} \quad y = x^2 + 2 \]

a. How many \( x \)-intercepts does each parabola have?
b. Can a parabola have more than two \( x \)-intercepts? Explain.
c. How many \( y \)-intercepts does each parabola have?
d. Can you draw a parabola with a different number of \( y \)-intercepts? Explain.

13. Consider the graph of \( y = x^2 - 3x - 10 \) shown at right.

a. Identify the \( x \)-intercepts from the graph.
b. Solve the quadratic equation \( x^2 - 3x - 10 = 0 \) by first factorising and then using the Null Factor Law.
c. Compare your answers for parts a and b. What do you notice?
d. Explain why you can use the graph to solve \( x^2 - 3x - 10 = 0 \). (Hint: what is the \( y \) value at each \( x \)-intercept?)

14. Solve each of these quadratic equations using the graphs in question 7.

\[ a \quad x^2 + 4x - 5 = 0 \quad b \quad x^2 - 4x = 0 \quad c \quad x^2 - 4 = 0 \]
\[ d \quad -x^2 - 4x + 5 = 0 \quad e \quad -x^2 + 4 = 0 \quad f \quad x^2 - 4x - 5 = 0 \]

15. If a parabola has \( x \)-intercepts at \((2, 0)\) and \((8, 0)\), what is the \( x \)-coordinate of the turning point?

16. If a parabola has only one \( x \)-intercept, at \((-4, 0)\), what is the \( x \)-coordinate of the turning point?

17. If an upright parabola has a turning point at \((-3, 1)\), how many \( x \)-intercepts does it have? How many \( y \)-intercepts does it have?

18. If an inverted parabola has a turning point at \((10, 0)\), how many \( x \)-intercepts does it have? How many \( y \)-intercepts does it have?

Reflect

How can you recognise a parabola?
4C Parabolas and transformations

Start thinking!

You can use the basic shape and features of a parabola to draw them without plotting points from a table. One way is to perform transformations such as dilation, reflection and translation on the basic parabola $y = x^2$ to produce the graph of another parabola.

1 a Plot the graph of $y = x^2$ on a Cartesian plane for $x$ values from $-3$ to $3$.
   b Identify the features of this parabola (type and coordinates of the turning point, equation of the axis of symmetry, $x$- and $y$-intercepts).

2 a On the same Cartesian plane as question 1, plot the graphs of:
   i $y = 2x^2$  
   ii $y = 3x^2$  
   iii $y = 4x^2$  
   iv $y = \frac{1}{2}x^2$  
   v $y = \frac{1}{4}x^2$.
   b Compare with the graph of $y = x^2$. Which features are the same? What makes each one different?
   c Which graphs are: i narrower than $y = x^2$? ii wider than $y = x^2$?
   d Explain how the coefficient of the $x^2$ term affects each graph.
   e Each parabola you drew in part a can be produced by dilating the graph of $y = x^2$. For example, the graph of $y = 2x^2$ is produced by dilating the graph of $y = x^2$ by a factor of 2 (made narrower). Describe the dilation performed to produce each of the other graphs.

3 Describe how you can recognise from its rule whether a parabola will be narrower or wider than the graph of $y = x^2$.

KEY IDEAS

- The graph of $y = x^2$ is an upright parabola with a minimum turning point at $(0, 0)$.
- The graph of $y = -x^2$ is an inverted parabola with a maximum turning point at $(0, 0)$. This graph is the reflection of the graph of $y = x^2$ in the $x$-axis.
- Transformations such as dilation, reflection and translation of the graph of $y = x^2$ are often used to produce graphs of other quadratic relationships.
- For rules of the form $y = ax^2$ where $a$ is positive, there is dilation only (dilation factor is $a$). For $0 < a < 1$ (coefficient between 0 and 1), the dilation produces a wider graph than $y = x^2$. For $a > 1$ (coefficient larger than 1), the dilation produces a narrower graph than $y = x^2$.
- For rules of the form $y = ax^2$ where $a$ is negative, there is dilation and reflection. For $-1 < a < 0$ (coefficient is between $-1$ and 0), the dilation produces a wider graph than $y = x^2$ which is reflected in the $x$-axis. For $a < -1$ (coefficient is less than $-1$), the dilation produces a narrower graph than $y = x^2$, which is reflected in the $x$-axis.
1 Consider the graphs of \( y = x^2 \), \( y = 2x^2 \) and \( y = \frac{1}{2}x^2 \). You may like to plot them on the same Cartesian plane.
   a Which parabola is wider than the graph of \( y = x^2 \)?
   b Write the rule for another parabola that is wider than the graph of \( y = x^2 \).
   c Which parabola is narrower than the graph of \( y = x^2 \)?
   d Write the rule for another parabola that is narrower than the graph of \( y = x^2 \).

**EXAMPLE 4C-1** Describing a transformation used to produce a graph from the graph of \( y = x^2 \)

For each rule below, describe the transformation needed to produce each graph from the graph of \( y = x^2 \).

<table>
<thead>
<tr>
<th>Rule</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 4x^2 )</td>
<td>The graph of ( y = x^2 ) is dilated by a factor of 4 to produce ( y = 4x^2 ). This will produce a narrower parabola.</td>
</tr>
<tr>
<td>( y = \frac{1}{3}x^2 )</td>
<td>The graph of ( y = x^2 ) is dilated by a factor of ( \frac{1}{3} ) to produce ( y = \frac{1}{3}x^2 ). This will produce a wider parabola.</td>
</tr>
</tbody>
</table>

2 For each rule below, describe the transformation needed to produce each graph from the graph of \( y = x^2 \).
   a \( y = 2x^2 \)
   b \( y = \frac{1}{2}x^2 \)
   c your rule from question 1b
   d your rule from question 1d

3 a Plot the graph of \( y = -x^2 \) on a Cartesian plane for \( x \) values from \(-3\) to \(3\).
   b Compare it to the graph of \( y = x^2 \). Has dilation been performed? Which transformation do you think has been performed?
   c Has the graph of \( y = x^2 \) been reflected in the \( x \)-axis or the \( y \)-axis to produce \( y = -x^2 \)? Explain.

4 a On the same Cartesian plane as question 3, plot the graphs of:
   i \( y = -2x^2 \)
   ii \( y = -3x^2 \)
   iii \( y = -4x^2 \)
   iv \( y = -\frac{1}{2}x^2 \)
   v \( y = -\frac{1}{4}x^2 \).
   b Compare each graph with the graph of \( y = -x^2 \). Which features are the same? What makes each one different?
   c Which graphs are:
      i narrower than \( y = -x^2 \)?
      ii wider than \( y = -x^2 \)?

**NOTE** If you drew graphs for question 1, you can use them to help you.
CHAPTER 4: NON-LINEAR RELATIONSHIPS

EXAMPLE 4C-2

Describing a transformation used to produce a graph from the graph of \( y = -x^2 \)

For each rule below, describe the transformation needed to produce each graph from the graph of \( y = -x^2 \).

\( a \) \( y = -6x^2 \)
\( b \) \( y = -\frac{1}{3}x^2 \)

**THINK**

\( a \) A dilation of factor 6 is needed. This will produce a narrower parabola.

\( b \) A dilation of factor \( \frac{1}{3} \) is needed. This will produce a wider parabola.

**WRITE**

\( a \) The graph of \( y = -x^2 \) is dilated by a factor of 6 to produce \( y = -6x^2 \). The graph of \( y = -6x^2 \) will be narrower than the graph of \( y = -x^2 \).

\( b \) The graph of \( y = -x^2 \) is dilated by a factor of \( \frac{1}{3} \) to produce \( y = -\frac{1}{3}x^2 \). The graph of \( y = -\frac{1}{3}x^2 \) will be wider than the graph of \( y = -x^2 \).

5 Describe the transformation needed to produce the graph for each rule below from the graph of \( y = -x^2 \).

\( a \) \( y = -2x^2 \)
\( b \) \( y = -\frac{1}{2}x^2 \)
\( c \) \( y = -\frac{1}{4}x^2 \)
\( d \) \( y = -4x^2 \)

**NOTE** Use your graph from question 4 to help you. Remember that you are comparing to the graph of \( y = -x^2 \).

EXAMPLE 4C-3

Identifying transformations to produce a graph from the graph of \( y = x^2 \)

For each rule below, identify the transformation/s needed to produce each graph from the graph of \( y = x^2 \). Describe the effect of the transformations.

\( a \) \( y = 3x^2 \)
\( b \) \( y = -\frac{1}{3}x^2 \)

**THINK**

\( a \) Look at the coefficient of the \( x^2 \) term. Since it is 3, a dilation of factor 3 is needed. This will produce a narrower upright parabola.

\( b \) Look at the coefficient of the \( x^2 \) term. Since it is \( -\frac{1}{3} \), a dilation of factor \( \frac{1}{3} \) is needed and then a reflection in the \( x \)-axis. This will produce a wider parabola that is inverted (turned upside down).

**WRITE**

\( a \) Dilation is needed. The graph of \( y = x^2 \) is dilated by a factor of 3 to produce \( y = 3x^2 \). The graph of \( y = 3x^2 \) will be narrower than the graph of \( y = x^2 \).

\( b \) Dilation and reflection are needed. The graph of \( y = x^2 \) is dilated by a factor of \( \frac{1}{3} \) to produce \( y = \frac{1}{3}x^2 \) and then reflected in the \( x \)-axis to produce \( y = -\frac{1}{3}x^2 \). The graph of \( y = -\frac{1}{3}x^2 \) will be wider than the graph of \( y = x^2 \) and reflected in the \( x \)-axis.
6 Identify the transformation/s needed to produce the graph of each rule below from the graph of \( y = x^2 \). Describe the effect of the transformations.

- \( a \) \( y = 5x^2 \)
- \( b \) \( y = -x^2 \)
- \( c \) \( y = -4x^2 \)
- \( d \) \( y = \frac{1}{4}x^2 \)
- \( e \) \( y = 10x^2 \)
- \( f \) \( y = -\frac{1}{7}x^2 \)
- \( g \) \( y = -8x^2 \)
- \( h \) \( y = -\frac{2}{3}x^2 \)

7 In question 4, there was more than one transformation performed on the graph of \( y = x^2 \) to produce each graph.

- \( a \) Which two transformations were used?
- \( b \) Describe the transformations performed on the graph of \( y = x^2 \) to produce each parabola.

8 Match each graph drawn on this Cartesian plane with its rule from the list provided below.

- \( A \) \( y = x^2 \)
- \( B \) \( y = -2x^2 \)
- \( C \) \( y = -\frac{1}{2}x^2 \)
- \( D \) \( y = \frac{1}{2}x^2 \)
- \( E \) \( y = 2x^2 \)
- \( F \) \( y = -x^2 \)

9 a Plot the graph of \( y = x^2 \) on a Cartesian plane for \( x \) values from -3 to 3.

b On the same Cartesian plane, plot the graphs of:

- \( i \) \( y = x^2 + 1 \)
- \( ii \) \( y = x^2 + 2 \)
- \( iii \) \( y = x^2 + 3 \)
- \( iv \) \( y = x^2 + 4 \)

c Compare each graph with the graph of \( y = x^2 \).

- \( i \) Which features are the same?
- \( ii \) What makes each one different?

d Explain how the constant term added to \( x^2 \) affects each graph.

e Each parabola you drew in part b can be produced by translating the graph of \( y = x^2 \). For example, the graph of \( y = x^2 \) is translated 1 unit up to produce the graph of \( y = x^2 + 1 \). Describe the translation performed to produce each of the other graphs.

10 a On the same Cartesian plane as question 9, plot the graphs of:

- \( i \) \( y = x^2 - 1 \)
- \( ii \) \( y = x^2 - 2 \)
- \( iii \) \( y = x^2 - 3 \)
- \( iv \) \( y = x^2 - 4 \).

b Compare each graph with the graph of \( y = x^2 \).

- \( i \) Which features are the same?
- \( ii \) What makes each one different?

c Explain how the constant term subtracted from \( x^2 \) affects each graph.

d Each parabola you drew in part a can be produced by translating the graph of \( y = x^2 \). For example, the graph of \( y = x^2 \) is translated 1 unit down to produce the graph of \( y = x^2 - 1 \). Describe the translation performed to produce each of the other graphs.
11 In questions 9 and 10, you looked at graphs with rules of the form \( y = x^2 + k \), where \( k \) could be any number.

a Describe how the value of \( k \) affects the graph of \( y = x^2 \).

b Explain how you know whether to move the graph of \( y = x^2 \) up or down.

c Copy and complete these sentences using the words up and down.
   When \( k \) is positive, the graph of \( y = x^2 \) is moved ________.
   When \( k \) is negative, the graph of \( y = x^2 \) is moved ________.

d What does it mean if \( k \) is 0?

12 For each rule below:
   i identify the value of \( k \), if the graph has the rule \( y = x^2 + k \)
   ii describe the transformation needed to produce each graph from the graph of \( y = x^2 \).

   a \( y = x^2 + 6 \)  b \( y = x^2 - 7 \)  c \( y = x^2 - 5 \)  d \( y = x^2 + 8 \)
   e \( y = x^2 + 9 \)  f \( y = x^2 - 11 \)  g \( y = x^2 + 1.5 \)  h \( y = x^2 - 7.2 \)

13 a Plot the graph of \( y = x^2 \) on a Cartesian plane for \( x \) values from −6 to 6.

b On the same Cartesian plane, plot the graphs of:
   i \( y = (x - 1)^2 \)  ii \( y = (x - 2)^2 \)
   iii \( y = (x - 3)^2 \)  iv \( y = (x - 4)^2 \)

c Compare each graph with the graph of \( y = x^2 \).
   i Which features are the same?
   ii What makes each one different?

d Explain how the constant term subtracted from \( x \) before squaring affects each graph.

e Each parabola you drew in part b can be produced by translating the graph of \( y = x^2 \). For example, the graph of \( y = x^2 \) is translated 1 unit right to produce the graph of \( y = (x - 1)^2 \). Describe the translation performed to produce each of the other graphs.

14 a On the same Cartesian plane as question 13, plot the graphs of:
   i \( y = (x + 1)^2 \)  ii \( y = (x + 2)^2 \)
   iii \( y = (x + 3)^2 \)  iv \( y = (x + 4)^2 \)

b Compare each graph with the graph of \( y = x^2 \).
   i Which features are the same?
   ii What makes each one different?

c Explain how the constant term added to \( x \) before squaring affects each graph.

d Each parabola you drew in part a can be produced by translating the graph of \( y = x^2 \). For example, the graph of \( y = x^2 \) is translated 1 unit left to produce the graph of \( y = (x + 1)^2 \). Describe the translation performed to produce each of the other graphs.
15 In questions 13 and 14, you looked at graphs with rules of the form \( y = (x - h)^2 \), where \( h \) could be any number.

a Describe how the value of \( h \) affects the graph of \( y = x^2 \).

b Explain how you know whether to move the graph of \( y = x^2 \) left or right.

c Copy and complete these sentences using the words left and right.

When \( h \) is positive, the graph of \( y = x^2 \) is moved _______.

When \( h \) is negative, the graph of \( y = x^2 \) is moved _______.

d What does it mean if \( h \) is 0?

16 For each rule below:

i identify the value of \( h \), if the graph has the rule \( y = (x - h)^2 \)

ii describe the transformation needed to produce each graph from the graph of \( y = x^2 \).

\[
\begin{align*}
a & : y = (x - 5)^2 \\
b & : y = (x + 7)^2 \\
c & : y = (x - 6)^2 \\
d & : y = (x + 9)^2 \\
e & : y = (x + 8)^2 \\
f & : y = (x - 12)^2 \\
g & : y = (x - 2.5)^2 \\
h & : y = (x + 6.7)^2
\end{align*}
\]

17 You can now perform more than one transformation on the graph of \( y = x^2 \).

a Plot the graph of each of these rules.

\[
\begin{align*}
i & : y = -x^2 + 2 \\
j & : y = -x^2 - 3 \\
k & : y = -(x - 3)^2 \\
l & : y = -(x + 2)^2 \\
m & : y = (x - 3)^2 + 2 \\
n & : y = -(x + 2)^2 - 3
\end{align*}
\]

b List the coordinates of the turning point for each graph.

c Identify whether each graph is upright or inverted.

d Match each description with one of the rules listed in part a.

A The graph of \( y = x^2 \) is reflected in the \( x \)-axis and then translated 3 units right.

B The graph of \( y = x^2 \) is reflected in the \( x \)-axis and then translated 2 units up.

C The graph of \( y = x^2 \) is translated 3 units right and 2 units up.

D The graph of \( y = x^2 \) is reflected in the \( x \)-axis and then translated 2 units left.

E The graph of \( y = x^2 \) is reflected in the \( x \)-axis and then translated 2 units left and 3 units down.

F The graph of \( y = x^2 \) is reflected in the \( x \)-axis and then translated 3 units down.

18 One of the general forms of a quadratic relationship is \( y = a(x - h)^2 + k \).

a Identify \( a, h \) and \( k \) for each rule in question 17a.

b Explain how the coordinates of the turning point can be worked out from the values of \( h \) and \( k \).

c Explain how the value of \( a \) affects whether the parabola is upright or inverted.

d Summarise your findings to explain how you think the values of \( a, h \) and \( k \) affect the graph of \( y = x^2 \). (Hint: describe the transformation that each value relates to.)

Reflect

Why is the graph of \( y = x^2 \) chosen as the basic parabola for transformations to be performed on to produce other parabolas?
4D Sketching parabolas using transformations

Start thinking!

Now that you can see the effect of transformations performed on the basic parabola of \( y = x^2 \), this knowledge can help you sketch a parabola without plotting points from a table.

1. Consider each quadratic relationship.
   - i. \( y = x^2 - 5 \)
   - ii. \( y = (x + 5)^2 \)
   - iii. \( y = -5x^2 \)
   - iv. \( y = -x^2 + 5 \)
   - v. \( y = -(x - 5)^2 \)

   a. Describe the transformation/s to be performed on the graph \( y = x^2 \) to produce the graph of each relationship.
   b. Will each parabola be upright or inverted? Explain how you can see this from the rule.
   c. Write the coordinates of the turning point of each parabola. Explain how you were able to do this.
   d. Use your answers from parts a–c to sketch the graph of each quadratic relationship.
   e. Which features did you use to sketch these graphs?

2. In your own words, explain how you can sketch a quadratic relationship without plotting points from a table.

KEY IDEAS

- Quadratic relationships can be written in the general form \( y = a(x - h)^2 + k \). This is known as the turning point form, as the coordinates of the turning point of the parabola can be easily identified as \((h, k)\).

- Transformations can be performed on the graph of \( y = x^2 \) to produce the sketch graph of \( y = a(x - h)^2 + k \).

  - dilation (narrower or wider)
    - For \( a > 0 \), upright parabola.
    - For \( a < 0 \), inverted parabola (reflection in \( x \)-axis).

  - vertical translation of \( k \) units
    - For \( k > 0 \), move up.
    - For \( k < 0 \), move down.

  - horizontal translation of \( h \) units
    - For \( h > 0 \), move right.
    - For \( h < 0 \), move left.

- You do not need to use grid paper to sketch a graph, as only the shape and important information is shown.
EXERCISE 4D  Sketching parabolas using transformations

1 For each sketch graph shown below:
   i identify whether the parabola is upright or inverted
   ii write the coordinates of the turning point
   iii describe how the graph may have been produced from $y = x^2$
   iv match each graph with its rule from the list shown at right.

   a
   b
   c

EXAMPLE 4D-1  Sketching a parabola by performing a vertical translation

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform a vertical translation to sketch the graph of $y = x^2 + 2$. Clearly show the coordinates of the turning point.

THINK

1 Identify the transformation. Vertical translation of 2 units up. (No dilation or reflection.)

2 Sketch the graph of $y = x^2$ and locate its turning point. Translate this point 2 units up. This becomes the turning point for $y = x^2 + 2$.

3 Use the position of the turning point at (0, 2) and the orientation of the parabola (upright) to sketch the graph.

WRITE

$y = x^2 + 2$

Graph of $y = x^2$ is translated 2 units up.

2 Sketch the graph of $y = x^2$ on a Cartesian plane, then perform a vertical translation to sketch the graph of each quadratic relationship. Clearly show the coordinates of the turning point on each parabola.

   a $y = x^2 + 3$
   b $y = x^2 + 1$
   c $y = x^2 - 2$
   d $y = x^2 + 6$
   e $y = x^2 - 4$
EXAMPLE 4D-2  Sketching a parabola by performing a horizontal translation

Sketch the graph of \( y = x^2 \) on a Cartesian plane, then perform a horizontal translation to sketch the graph of \( y = (x - 4)^2 \). Clearly show the coordinates of the turning point.

**THINK**

1. Identify the transformation. Horizontal translation of 4 units right. (No dilation or reflection.)

2. Sketch the graph of \( y = x^2 \) and locate its turning point. Translate this point 4 units right. This becomes the turning point for \( y = (x - 4)^2 \).

3. Use the position of the turning point at \((4, 0)\) and the orientation of the parabola (upright) to sketch the graph.

**WRITE**

\[ y = (x - 4)^2 \]
Graph of \( y = x^2 \) is translated 4 units right.

---

3. Sketch the graph of \( y = x^2 \) on a Cartesian plane, then perform a horizontal translation to sketch the graph of each quadratic relationship. Clearly show the coordinates of the turning point on each parabola.

   \( a \)  \( y = (x - 3)^2 \)
   \( b \)  \( y = (x - 1)^2 \)
   \( c \)  \( y = (x + 2)^2 \)
   \( d \)  \( y = (x + 4)^2 \)
   \( e \)  \( y = (x - 5)^2 \)

4. Look at the graph shown at right.
   
   \( a \)  Is the parabola upright or inverted?
   \( b \)  Identify the coordinates of the turning point.
   \( c \)  Which of these rules would best match the graph? Explain.
   
   \( A \)  \( y = (x - 2)^2 + 4 \)
   \( B \)  \( y = -(x - 4)^2 - 2 \)
   \( C \)  \( y = (x - 4)^2 - 2 \)
   \( D \)  \( y = (x + 4)^2 - 2 \)

5. Compare \( y = (x - 3)^2 + 4 \) to the turning point form of a quadratic relationship.
   
   \( a \)  Identify \( a, h \) and \( k \).
   \( b \)  What information can you identify from the values of \( a, h \) and \( k \)?
   \( c \)  Use \( (h, k) \) to write the coordinates of the turning point.
   \( d \)  Explain how \( (h, k) \) is related to the translations performed on \( y = x^2 \) to produce \( y = (x - 3)^2 + 4 \).

6. Repeat question 5 for the quadratic relationship you identified in question 4c. Show that the information you obtain matches the graph provided in question 4c.
EXAMPLE 4D-3 Sketching a parabola by performing more than one transformation

Sketch the graph of \( y = (x + 2)^2 + 1 \) by performing transformations on \( y = x^2 \).
Clearly show the coordinates of the turning point.

THINK

1 Identify the transformations. Horizontal translation of 2 units left and vertical translation of 1 unit up. (No dilation or reflection.) Alternatively, compare to the turning point form of a quadratic, \( y = a(x - h)^2 + k \), to identify the transformations. \( a = 1 \) (upright parabola, same shape as \( y = x^2 \)), \( h = -2 \) (move 2 units left) and \( k = 1 \) (move 1 unit up).

2 Identify the type and position of the turning point. General coordinates are \((h, k)\).

3 Use the position of the turning point and the orientation of the parabola (upright) to sketch the graph.

WRITE

\[ y = (x + 2)^2 + 1 \]
\[ y = [x - (-2)]^2 + 1 \]
Graph of \( y = x^2 \) is translated 2 units left and 1 unit up.
Minimum turning point at \((-2, 1)\).

7 Sketch the graph of each quadratic relationship by performing transformations on \( y = x^2 \). Clearly show the coordinates of the turning point on each parabola.

- \( a \) \( y = (x - 2)^2 + 3 \)
- \( b \) \( y = (x - 1)^2 - 2 \)
- \( c \) \( y = (x + 4)^2 + 6 \)
- \( d \) \( y = (x + 5)^2 - 4 \)
- \( e \) \( y = (x - 7)^2 - 5 \)

8 Perform a reflection and then a translation on \( y = x^2 \) to sketch the graph of each quadratic relationship. Show the coordinates of the turning point on each parabola.

- \( a \) \( y = -(x - 2)^2 \)
- \( b \) \( y = -x^2 + 4 \)
- \( c \) \( y = -(x + 6)^2 \)
- \( d \) \( y = -x^2 - 3 \)
- \( e \) \( y = -(x - 1)^2 \)

9 Match each graph with its rule from the list provided at right.

- \( a \) \( y = (x - 3)^2 + 2 \)
- \( b \) \( y = -(x - 3)^2 + 2 \)
- \( c \) \( y = (x + 3)^2 - 2 \)
- \( d \) \( y = -(x + 3)^2 + 2 \)
- \( e \) \( y = (x - 2)^2 - 3 \)
- \( f \) \( y = -(x + 2)^2 + 3 \)
10 Answer the questions below for each parabola.

a \[ y = (x - 1)^2 - 4 \]

b \[ y = -(x + 3)^2 + 1 \]

c \[ y = (x - 3)^2 - 9 \]

i Is the parabola upright or inverted?

ii What are the coordinates of the turning point?

iii What are the coordinates of the $y$-intercept and the $x$-intercepts?

11 Write the rule for each from the given information. Assume each parabola has the same shape as $y = x^2$; that is, no dilation has been performed.

a upright, turning point at (3, 7)  

b upright, turning point at (−2, 5)  

c inverted, turning point at (2, 4)  

d inverted, turning point at (6, −1)  

e upright, turning point at (9, 0)  

f inverted, turning point at (0, 4)  

g inverted, turning point at (−1, −2)  

h upright, turning point at (0, −5)

12 Sketch each quadratic relationship after completing these steps:

i describe the transformation/s to be performed on the graph of $y = x^2$

ii identify whether the parabola will be upright or inverted

iii write the coordinates of the turning point.

a \[ y = x^2 + 4 \]  

b \[ y = -(x - 8)^2 \]  

c \[ y = (x - 3)^2 - 2 \]  

d \[ y = -(x + 4)^2 + 5 \]  

e \[ y = -4x^2 \]  

f \[ y = -x^2 - 1 \]  

g \[ y = (x + 7)^2 \]  

h \[ y = (x + 1)^2 - 3 \]

13 A quadratic relationship has the rule \[ y = (x - 4)^2 + 5. \]

a What are the coordinates of the turning point?

b What is the smallest $y$ value that this relationship can have?

14 A quadratic relationship has the rule \[ y = -(x + 1)^2 + 2. \] What is the largest $y$ value that this relationship can have? Explain.

15 Jenna throws a basketball to a teammate. The height of the ball can be represented by the relationship \[ h = -(t - 2)^2 + 6, \] where $h$ is the height in metres after $t$ seconds.

a What are the coordinates of the turning point of this relationship.

b What is the value of $h$ when:  

i $t = 0$?  

ii $t = 4$?

c Sketch the graph of this relationship from $t = 0$ to $t = 4$.

d Use the graph to find:

i the height at which the ball left Jenna’s hands

ii the maximum height of the ball during the pass to her teammate.
16 An amateur golfer hits a ball up into the air. The path of the ball follows the relationship \( y = -(x - 10)^2 + 100 \), where \( y \) is the height of the ball for a horizontal distance \( x \) from where the ball was hit. Both \( x \) and \( y \) are in metres.

a Sketch the graph of this relationship from when the ball was hit to when it landed. (Hint: use the fact that a parabola is symmetrical.)

b What was the maximum height of the golf ball?

c How far from the golfer did the ball land?

17 For each quadratic relationship, identify:

i whether the graph will be narrower or wider than the graph of \( y = x^2 \)

ii whether the parabola will be upright or inverted

iii the coordinates of the turning point.

\[
\begin{align*}
a & \quad y = 2(x - 4)^2 - 3 \\
b & \quad y = -3(x + 1)^2 + 5 \\
c & \quad y = 4(x + 2)^2 \\
d & \quad y = -5x^2 - 4 \\
e & \quad y = -(x - 5)^2 + 4 \\
f & \quad y = \frac{1}{2}(x + 2)^2 + 6 \\
g & \quad y = -\frac{1}{3}(x - 3)^2 - 4 \\
h & \quad y = \frac{1}{2}x^2 + 3 \\
i & \quad y = -\frac{1}{2}(x - 7)^2 \\
\end{align*}
\]

18 Sketch the graph of each quadratic relationship on the same Cartesian plane. Clearly show the coordinates of the turning point.

\[
\begin{align*}
a & \quad y = 3(x - 2)^2 + 4 \\
b & \quad y = \frac{1}{2}(x + 4)^2 + 1 \\
c & \quad y = -2(x + 1)^2 - 3 \\
\end{align*}
\]

19 The height above the ground for a bungee jumper is measured from the start of the first downward movement to just before the start of the second downward movement. These measurements form the relationship \( h = 5(t - 4)^2 + 10 \), where \( h \) is the height in metres after time \( t \) seconds.

a Sketch a graph of the relationship.

b How high is the person off the ground at the start of the jump?

c What is the lowest height above the ground the person falls to in the first downward movement of the jump?

20 Explain why the graph of \( y = -2x^2 \) is a reflection in the \( x \)-axis of \( y = 2x^2 \).

21 Write the rule of the parabola that is the reflection in the \( x \)-axis of the graph with each rule below.

\[
\begin{align*}
a & \quad y = 4x^2 \\
b & \quad y = -\frac{1}{3}x^2 \\
c & \quad y = -(x + 2)^2 \\
d & \quad y = x^2 + 5 \\
e & \quad y = -(x + 1)^2 - 4 \\
f & \quad y = 2(x - 5)^2 - 3 \\
\end{align*}
\]

22 Write a rule for the parabola produced after performing each set of transformations on the graph of \( y = x^2 \).

a dilation by a factor of 3 then a translation of 2 units right

b reflection in the \( x \)-axis then a translation of 4 units down and 5 units left

c dilation by a factor of \( \frac{1}{2} \) and a reflection in the \( x \)-axis

d dilation by a factor of 4, reflection in the \( x \)-axis then a translation of 2 units left

\[ \text{Reflect} \]

How is writing a quadratic relationship in turning point form useful when sketching its graph?
4E Sketching parabolas using intercepts

Start thinking!
1. Explain why transformations are easy to use for a quadratic rule like $y = (x + 3)^2 + 4$ but not as easy for one like $y = x^2 + 4x + 3$.

Another way to identify information to sketch a parabola is to find the x- and y-intercepts.

Consider sketching the graph of $y = x^2 + 4x + 3$.

2. a. What is the x-coordinate at the y-intercept of any graph?
   b. Substitute this value for $x$ into the rule $y = x^2 + 4x + 3$ and simplify.
   c. What is the y-intercept?

3. a. What is the y-coordinate at the x-intercepts of any graph?
   b. Substitute this value for $y$ into the rule $y = x^2 + 4x + 3$.
   c. Solve the equation to find the value of $x$.
   d. What are the x-intercepts?

4. List the coordinates of the three points you can use to help sketch the graph of $y = x^2 + 4x + 3$.

5. Look at the coefficient of the $x^2$ term. Will the parabola be upright or inverted?

6. Plot the three points and draw a parabola through them to produce a sketch graph of $y = x^2 + 4x + 3$.
   Label your graph with its rule.

7. How could you work out the coordinates of the turning point?

8. In your own words, explain how to sketch a quadratic relationship in the form $y = ax^2 + bx + c$ using intercepts.

KEY IDEAS

► One way of sketching a quadratic relationship is to use the x- and y-intercepts. The coordinates of the turning point and the orientation of the parabola (upright or inverted) can also be identified.

► The x-intercept/s are found by substituting $y = 0$ into the rule and solving for $x$. The equation may need to be factorised first so that the Null Factor Law can be used to solve the quadratic equation. A parabola can have two, one or no x-intercepts.

► The y-intercept is found by substituting $x = 0$ into the rule and simplifying.

► The axis of symmetry of a parabola is midway between the x-intercepts. Hence, the x-coordinate of the turning point is halfway between the $x$ values at the x-intercepts. The y-coordinate of the turning point is found by substituting the x-coordinate into the rule and simplifying.
**EXAMPLE 4E-1** Finding coordinates of the x- and y-intercepts of a quadratic relationship

Find the coordinates of the x- and y-intercepts for \( y = x^2 - 6x \).

**THINK**

1. To find the x-intercept/s, substitute \( y = 0 \) into the rule. (You may like to swap the sides of the equation.)

2. Factorise the quadratic expression on the left side.

3. Solve the equation using the Null Factor Law.

4. Write the coordinates of the x-intercepts.

5. To find the y-intercept, substitute \( x = 0 \) into the rule and simplify.

6. Write the coordinates of the y-intercept.

**WRITE**

\[ y = x^2 - 6x \]

\[ x \text{-intercepts: when } y = 0, \]

\[ 0 = x^2 - 6x \]

\[ x^2 - 6x = 0 \]

\[ x(x - 6) = 0 \]

\[ x = 0 \text{ or } x - 6 = 0 \]

\[ x = 0 \text{ or } x = 6 \]

Coordinates of the x-intercepts are \((0, 0)\) and \((6, 0)\).

\[ y \text{-intercept: when } x = 0, \]

\[ y = 0 - 0 = 0 \]

Coordinates of the y-intercept are \((0, 0)\).

---

**EXERCISE 4E** Sketching parabolas using intercepts

1. Copy and complete the given working to find the coordinates of the x- and y-intercepts for each quadratic relationship.

   a. \( y = x^2 - 2x - 15 \)

   \[ x \text{-intercepts: when } y = __, \]

   \[ __ = x^2 - 2x - 15 \]

   \[ x^2 - 2x - 15 = __ \]

   \[ (x + __)(x - __) = __ \]

   \[ x + __ = 0 \text{ or } x - __ = 0 \]

   \[ x = __ \text{ or } x = __ \]

   Coordinates of the x-intercepts are \((__, __)\) and \((__, __)\).

   \[ y \text{-intercepts: when } x = __, \]

   \[ y = __ - __ - __ = __ \]

   Coordinates of the y-intercept are \((__, __)\).

   b. \( y = x^2 - 1 \)

   \[ x \text{-intercepts: when } y = __, \]

   \[ 0 = x^2 - 1 \]

   \[ x^2 - 1 = __ \]

   \[ (x + __)(x - __) = __ \]

   \[ x + __ = 0 \text{ or } x - __ = 0 \]

   \[ x = __ \text{ or } x = __ \]

   Coordinates of the x-intercepts are \((__, __)\) and \((__, __)\).

   \[ y \text{-intercepts: when } x = __, \]

   \[ y = __ - __ = __ \]

   Coordinates of the y-intercept are \((__, __)\).
For each quadratic relationship, find the coordinates of:

- **i** the \(x\)-intercepts
- **ii** the \(y\)-intercept.

\[
a. \quad y = x^2 - 2x \\
b. \quad y = x^2 + 8x \\
c. \quad y = x^2 + 6x + 8 \\
d. \quad y = x^2 - 8x + 12 \\
e. \quad y = x^2 - 4x - 5 \\
f. \quad y = x^2 - 9
\]

**EXAMPLE 4E-2** Finding coordinates of the turning point using \(x\)-intercepts

Find the coordinates of the turning point for \(y = x^2 - 6x\).

**THINK**

1. Find the \(x\)-intercepts (see Example 4E-1).

2. Since a parabola is symmetrical, the \(x\)-coordinate of the turning point is halfway between the \(x\)-intercepts. Alternatively, find the average of the two \(x\) values.

3. Find the \(y\)-coordinate of the turning point by substituting \(x = 3\) into the rule and simplifying.

4. Write the coordinates of the turning point.

**WRITE**

\[
y = x^2 - 6x \\
\text{\(x\)-intercepts are 0 and 6.} \\
\text{Halfway between 0 and 6 is 3,} \\
or \quad x = \frac{0 + 6}{2} = 3. \\
\text{When } x = 3, \\
y = 3^2 - 6 \times 3 \\
= 9 - 18 \\
= -9 \\
\text{Coordinates of the turning point} \\
\text{are (3, } -9\text{).}
\]

3. Copy and complete the given working to find the coordinates of the turning point for each quadratic relationship. (Hint: refer to your answers for question **1**.)

\[
a. \quad y = x^2 - 2x - 15 \\
\text{\(x\)-intercepts are } -3 \text{ and } _. \\
\text{Halfway between } -3 \text{ and } _= _, \\
or \quad x = -3 + \_ = _. \\
\text{When } x = _, \\
y = _^2 - 2 \times _ - 15 \\
= _ - _ - 15 \\
= _ \\
\text{Coordinates of the turning point} \\
\text{are (_, _.).}
\]

\[
b. \quad y = x^2 - 1 \\
\text{\(x\)-intercepts are } -1 \text{ and } _. \\
\text{Halfway between } -1 \text{ and } _= _, \\
or \quad x = -1 + \_ = _. \\
\text{When } x = _, \\
y = _^2 - _. \\
= _ - _ \\
= _ \\
\text{Coordinates of the turning point} \\
\text{are (_, _.).}
\]

4. Find the coordinates of the turning point for each quadratic relationship in question **2**.

5. Sketch the graph of each quadratic relationship in question **2** using your answers to questions **2** and **4**.
Sketching a parabola using x- and y-intercepts

**THINK**

1. Find the x-intercepts by substituting y = 0 into the rule and solving for x. Factorise so that the Null Factor Law can be used.

2. Find the y-intercept by substituting x = 0 into the rule and simplifying.

3. Find the coordinates of the turning point. The x-coordinate is halfway between the x-intercepts (or the average of the two x values).

4. Plot the points for the x- and y-intercepts and the turning point on a Cartesian plane.

5. Draw an upright parabola through the points and label with the rule. (The parabola is upright since the coefficient of the $x^2$ term is positive.)

**WRITE**

Sketch the graph of $y = x^2 + 2x - 8$ using intercepts. Label the turning point with its coordinates.

- When $y = 0$, $x^2 + 2x - 8 = 0$.
- $(x + 4)(x - 2) = 0$.
- $x = -4$ or $x = 2$.
- x-intercepts are -4 and 2.

- When $x = 0$, $y = 0^2 + 2 \times 0 - 8 = -8$.
- y-intercept is -8.

- At turning point, $x = \frac{-4 + 2}{2} = -1$.
- When $x = -1$,
  - $y = (-1)^2 + 2 \times (-1) - 8 = -9$.
- Coordinates of turning point are $(−1, −9)$.

6. Sketch the graph of each quadratic relationship using intercepts. Label the turning point with its coordinates.

   a. $y = x^2 - 6x + 5$
   b. $y = x^2 + 4x - 12$
   c. $y = x^2 - 2x - 3$
   d. $y = x^2 - 4$
   e. $y = x^2 + 4x$
   f. $y = x^2 + 2x - 15$
   g. $y = x^2 - 6x - 7$
   h. $y = x^2 - 5x$

7. Match each graph with its rule from the list below.

   - $y = x^2 + x - 6$
   - $y = -x^2 - x + 6$
   - $y = x^2 - x - 6$

8. Explain how you can tell whether a parabola will be upright or inverted from its rule. Use your answers to question 7 as examples in your explanation.
9 Consider the graphs of \( y = (x + 3)(x - 2) \) and \( y = -(x + 3)(x - 2) \).
   a Find the \( x \)-intercepts for each graph.
   b Explain why the graphs will be different even though each parabola has the same \( x \)-intercepts.

10 For each quadratic relationship:
   i identify whether its graph will be an upright or inverted parabola
   ii find the coordinates of the \( x \)- and \( y \)-intercepts
   iii find the coordinates of the turning point
   iv sketch its graph.
   a \( y = (x + 5)(x - 3) \)
   b \( y = -(x + 5)(x - 3) \)
   c \( y = -x(x + 4) \)
   d \( y = x^2 + 4x \)
   e \( y = x^2 + 8x + 12 \)
   f \( y = -x^2 - 8x - 12 \)
   g \( y = x^2 - 16 \)
   h \( y = 16 - x^2 \)
   i \( y = -x^2 + 6x + 7 \)
   j \( y = x^2 - 6x - 7 \)
   k \( y = x^2 + 3x + 2 \)
   l \( y = x^2 - x - 6 \)

11 For each quadratic relationship below:
   i identify whether its graph will be an upright or inverted parabola
   ii find the coordinates of the \( x \)- and \( y \)-intercepts
   iii identify the coordinates of the turning point
   iv sketch its graph.
   a \( y = (x - 2)^2 - 1 \)
   b \( y = -(x + 1)^2 + 9 \)
   c \( y = -(x + 4)^2 + 1 \)
   d \( y = (x - 3)^2 - 4 \)
   e \( y = -(x + 2)^2 \)
   f \( y = x^2 + 8x + 16 \)
   g \( y = x^2 - 4x + 4 \)
   h \( y = -x^2 - 6x - 9 \)

12 Consider the graph shown at right.
   a Is the parabola upright or inverted?
   b How many \( y \)-intercepts does the parabola have? List the coordinates of the \( y \)-intercept/s.
   c How many \( x \)-intercepts does the parabola have? List the coordinates of the \( x \)-intercept/s.
   d What are the coordinates of the turning point?

13 Sketch the graph of each quadratic relationship using intercepts.
   Label the turning point with its coordinates.
   a \( y = (x - 1)^2 \)
   b \( y = -(x + 2)^2 \)
   c \( y = x^2 + 8x + 16 \)
   d \( y = x^2 - 4x + 4 \)
   e \( y = -x^2 - 6x - 9 \)

14 Consider the graph shown at right.
   a Is the parabola upright or inverted?
   b How many \( y \)-intercepts does the parabola have? List the coordinates of the \( y \)-intercept/s.
   c How many \( x \)-intercepts does the parabola have? List the coordinates of the \( x \)-intercept/s.
   d What are the coordinates of the turning point?
15 Sketch the graph of each quadratic relationship using intercepts. Label the turning point with its coordinates.

\[ a \quad y = (x - 3)^2 + 1 \quad b \quad y = (x + 2)^2 + 3 \quad c \quad y = -(x - 1)^2 - 4 \]

\[ d \quad y = (x + 4)^2 + 1 \quad e \quad y = -(x + 3)^2 - 2 \]

16 Rhys fires an arrow from a bow. The position of the arrow can be represented by the quadratic relationship \( h = -0.1(d + 1)(d - 15) \) where \( h \) is the height above the ground and \( d \) is the horizontal distance from where the arrow was fired. Both \( h \) and \( d \) are in metres.

\[ a \] Sketch the graph of this relationship by finding the intercepts.

\[ b \] How high does the arrow reach?

\[ c \] At what height off the ground was the arrow fired?

\[ d \] What horizontal distance did the arrow fly before hitting the ground?

17 The amount of money in Teresa’s bank account over a 3-week interval can be represented by the quadratic relationship \( a = t^2 - 20t + 84 \), where \( a \) is the account balance in dollars after \( t \) days.

\[ a \] Sketch the graph of this relationship for the 3-week interval.

\[ b \] How much money was in Teresa’s account at the start of the 3 weeks?

\[ c \] How much money was in her account after 2 days?

\[ d \] When was Teresa’s account first overdrawn?

\[ e \] What was the highest amount that she owed the bank during the 3 weeks?

\[ f \] When was her account balance back to zero?

\[ g \] What was the highest amount in Teresa’s account over the 3 weeks?

18 A soccer ball is kicked off the ground. Its path can be represented by the quadratic relationship \( y = -0.2x^2 + 2.4x \), where \( x \) is the horizontal distance in metres and \( y \) is the vertical distance in metres.

\[ a \] Sketch the graph of this relationship by finding the intercepts.

\[ b \] What was the maximum height of the soccer ball?

\[ c \] What horizontal distance had the soccer ball travelled when it was at its maximum height?

\[ d \] What horizontal distance did the soccer ball travel before hitting the ground?

19 For each set of \( x \)-intercepts, write a rule for a parabola that would match.

\[ a \quad x = 2 \quad b \quad x = 7 \quad c \quad x = 0 \quad d \quad x = 8 \quad e \quad x = -4 \quad f \quad x = 5 \]

20 Write another two quadratic rules for each set of \( x \)-intercepts in question 19.

Reflect

How many \( x \)-intercepts and \( y \)-intercepts does a parabola have?
**4F Circles and other non-linear relationships**

**Start thinking!**

1. Look at the three circles shown.
   a. Write the coordinates of the centre of each circle.
   b. Identify the radius of each circle.
   c. Compare the features of each circle with its rule. Can you see any patterns? Explain.

2. Similar to parabolas, you can perform transformations on the basic graph of a circle with centre at (0, 0). Let’s consider translations left or right and up or down. Look at each circle.
   a. Write the coordinates of the centre of each circle.
   b. By comparing the position of the centre of the circle, describe the translation that has been performed on the basic graph of $x^2 + y^2 = 4$ to produce each circle.
   c. Compare the features of each circle with its rule. Can you see any patterns? Explain.
   d. Can you sketch the graph of $(x - 1)^2 + (y - 3)^2 = 4$? Try it. Explain your reasoning.

**KEY IDEAS**

- Relationships for circles can be written in the general form $(x - h)^2 + (y - k)^2 = r^2$, where $(h, k)$ are the coordinates of the centre of the circle and $r$ is the radius.

- The relationship for a circle with centre at (0, 0) and radius $r$ is written as $x^2 + y^2 = r^2$.

- Translations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the sketch graph of $(x - h)^2 + (y - k)^2 = r^2$.

- **Horizontal translation of** $h$ units: For $h > 0$, move right. For $h < 0$, move left.
- **Vertical translation of** $k$ units: For $k > 0$, move up. For $k < 0$, move down.
EXERCISE 4F  Circles and other non-linear relationships

1 For each rule, identify the coordinates of the centre of the circle and the radius.
   a  \( x^2 + y^2 = 36 \)  b  \( x^2 + y^2 = 64 \)  c  \( x^2 + y^2 = 1 \)  d  \( x^2 + y^2 = 100 \)

2 Consider the circle with the rule \( x^2 + y^2 = 9 \).
   a What are the coordinates of the centre of this circle?
   b What is the radius?
   c To sketch this circle easily, you need to mark five points on the Cartesian plane. What do you think these five points will be? Discuss this with a classmate.
   d On a Cartesian plane, mark a point for the centre of this circle. Use the radius to mark a point directly above, below, left and right of the centre. List the coordinates of the four points that sit on the circumference of the circle.
   e Draw a circle through these four points. You may like to use a pair of compasses to help you. On the scale of the axes, indicate the highest and lowest values of \( x \) and \( y \) for the circle.

3 Sketch a circle on the Cartesian plane with centre at \((0, 0)\) and radius of 4 units. Write the rule for this circle.

4 Write the rule for a circle with centre at \((0, 0)\) and radius of 7 units.

5 Consider the circle with rule \((x - 2)^2 + (y - 5)^2 = 9\). Compare this to the general rule for a circle \((x - h)^2 + (y - k)^2 = r^2\).
   a Identify \(h, k\) and \(r\).
   b Use your answers to part a to identify the coordinates of the centre of the circle.
   c What is the radius of the circle?

6 Repeat question 5 for each of these rules.
   i  \((x - 1)^2 + (y - 4)^2 = 16\)  ii  \((x - 6)^2 + (y + 3)^2 = 25\)
   iii  \((x + 4)^2 + (y - 2)^2 = 1\)  iv  \((x + 7)^2 + (y + 2)^2 = 36\)

7 Consider this circle drawn on the Cartesian plane.
   a What are the coordinates of the centre of the circle?
   b What is the radius of the circle?
   c Explain how the coordinates of the centre of the circle and the radius can be read from the rule if it is written as \((x + 2)^2 + (y - 0)^2 = 7^2\).
   d Describe the translation/s that have been performed on the basic graph of \(x^2 + y^2 = 49\) to produce the circle.
8 Consider each circle drawn on the Cartesian plane.

\[
\text{i} \quad (x - 3)^2 + (y + 1)^2 = 25 \\
\text{ii} \quad (x + 4)^2 + (y + 2)^2 = 16
\]

a What are the coordinates of the centre of each circle?
b What is the radius of each circle?
c Describe the translation/s that have been performed on the basic graph of \(x^2 + y^2 = r^2\) to produce each circle.

**EXAMPLE 4F-1** Sketching a circle from its rule

Identify the coordinates of the centre and the radius of the circle with the rule \((x + 2)^2 + (y - 4)^2 = 9\) and hence sketch its graph.

**THINK**

1 Identify any translations of the graph of \(x^2 + y^2 = 9\). Horizontal translation of 2 units left and vertical translation of 4 units up. Alternatively, compare to \((x - h)^2 + (y - k)^2 = r^2\), to identify the translations. \(h = -2\) (move 2 units left) and \(k = 4\) (move 4 units up).

2 Identify the coordinates of the centre. Perform translations on \((0, 0)\) or use \((h, k)\).

3 Identify the radius: \(r^2 = 9\) so \(r = 3\).

4 Sketch the graph by first marking the centre of the circle at \((-2, 4)\) and identifying four points that are 3 units above, below, left and right of the centre. These four points are: \((-2, 7), (-2, 1), (-5, 4)\) and \((1, 4)\).

**WRITE**

\((x + 2)^2 + (y - 4)^2 = 9\)

The graph of \(x^2 + y^2 = 9\) is translated 2 units left and 4 units up.

centre at \((-2, 4)\)

radius of 3 units

9 Identify the coordinates of the centre and the radius of each circle with these rules and hence sketch its graph.

\[
a \quad (x - 2)^2 + (y - 3)^2 = 4 \\
b \quad (x - 1)^2 + (y - 5)^2 = 9 \\
c \quad (x + 3)^2 + (y - 2)^2 = 36 \\
d \quad (x - 4)^2 + (y + 3)^2 = 25 \\
e \quad x^2 + y^2 = 1 \\
f \quad (x - 6)^2 + y^2 = 4 \\
g \quad x^2 + (y + 4)^2 = 49 \\
h \quad (x + 5)^2 + (y + 1)^2 = 16
\]

10 Produce each graph in question 9 using digital technology. Compare your answers.
### Example 4F-2 Writing the rule for a circle

Write the rule for a circle with radius of 6 units and centre at (2, –5).

<table>
<thead>
<tr>
<th>THINK</th>
<th>WRITE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Write the general rule for a circle.</td>
<td>Rule for circle with radius ( r ) and centre at ((h, k)) is ((x - h)^2 + (y - k)^2 = r^2).</td>
</tr>
<tr>
<td>2. Identify ( h ), ( k ), and ( r ).</td>
<td>( h = 2, k = -5 ) and ( r = 6 )</td>
</tr>
<tr>
<td>3. Substitute for ( h ), ( k ), and ( r ) in the general rule and simplify.</td>
<td>((x - 2)^2 + (y + 5)^2 = 36)</td>
</tr>
</tbody>
</table>

### 11 Write the rule for each of these circles using the information provided.

- **a** circle with radius of 4 units and centre at (3, 5)
- **b** circle with radius of 5 units and centre at (–2, 4)
- **c** circle with radius of 9 units and centre at (–7, –6)
- **d** circle with radius of 11 units and centre at (4, –8)

### 12 Identify the centre and the radius of each circle and hence write its rule.

- **a**
  - Centre: (2, 4)
  - Radius: 4 units
  - Rule: \((x - 2)^2 + (y - 4)^2 = 16\)

- **b**
  - Centre: (–3, 3)
  - Radius: 5 units
  - Rule: \((x + 3)^2 + (y - 3)^2 = 25\)

- **c**
  - Centre: (–7, –6)
  - Radius: 9 units
  - Rule: \((x + 7)^2 + (y + 6)^2 = 81\)

- **d**
  - Centre: (4, –8)
  - Radius: 11 units
  - Rule: \((x - 4)^2 + (y + 8)^2 = 121\)

### 13 Sketch a circle on the Cartesian plane with centre at (2, 5) and radius of 4 units. Write the rule for this circle.

### 14 An unusual circular running track is mapped on to a Cartesian plane using the relationship \((x - 30)^2 + (y - 40)^2 = 2500\). All measurements are in metres.

- **a** Sketch the graph of this relationship.
- **b** Calculate the length of the running track to the nearest metre. (Hint: the formula for the circumference of a circle is \( C = 2\pi r \).)
- **c** The surface of the ground inside the running track is to be sown with grass seed. To the nearest square metre, what area is to be sown? (Hint: the formula for the area of a circle is \( A = \pi r^2 \).)
15 There are many other non-linear relationships. Let’s look at a basic cubic relationship.

a Plot the graph of \( y = x^3 \) after completing this table of values. Draw a smooth line through the points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -3 )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( -27 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Describe the shape of the graph.

c An important feature of the graph of \( y = x^3 \) is the point of inflection at \((0, 0)\). Mark this on your graph.

d The sketch graphs of \( y = x^3 \) and \( y = x^3 + 2 \) are shown at right.

i Identify the coordinates of the point of inflection for \( y = x^3 + 2 \).

ii Describe how you could use a translation to produce the graph of \( y = x^3 + 2 \) from the graph of \( y = x^3 \).

e Use your understanding of translations to describe how the graphs of these cubic relationships can be produced from the graph of \( y = x^3 \).

i \( y = x^3 + 1 \)   
ii \( y = x^3 - 3 \)   
iii \( y = (x - 2)^3 \)   
iv \( y = (x + 4)^3 \)   
v \( y = (x - 1)^3 + 2 \)

f Use your answers to part e to sketch each relationship. Clearly show the coordinates of the point of inflection on each graph.

g Use a calculator or other digital technology to produce the same graphs.

16 Consider a relationship involving the square root of \( x \).

a Plot the graph of \( y = \sqrt{x} \) after completing this table of values. Draw a smooth line through the points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 4 )</th>
<th>( 9 )</th>
<th>( 16 )</th>
<th>( 25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( 0 )</td>
<td>( 3 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Describe the shape of the graph. Can you plot points for negative values of \( x \)? Explain.

c An important feature of the graph of \( y = \sqrt{x} \) is the point on the graph where \( y \) is a minimum. The coordinates of this point are \((0, 0)\). Mark this point on your graph.

d The sketch graphs of \( y = \sqrt{x} \) and \( y = \sqrt{x - 3} \) are shown at right.

i Identify the coordinates of the point on the graph of \( y = \sqrt{x - 3} \) where \( y \) is a minimum.

ii Describe how you could use a translation to produce the graph of \( y = \sqrt{x - 3} \) from the graph of \( y = \sqrt{x} \).

e Use your understanding of translations to describe how the graphs of these relationships can be produced from the graph of \( y = \sqrt{x} \).

i \( y = \sqrt{x} + 2 \)   
ii \( y = \sqrt{x} - 1 \)   
iii \( y = \sqrt{x - 1} \)   
iv \( y = \sqrt{x + 4} \)   
v \( y = \sqrt{x - 2} + 3 \)

f Use your answers to part e to sketch each relationship. Clearly show the coordinates of the point on each graph where \( y \) is a minimum.

g Use a calculator or other digital technology to produce the same graphs.
17 Consider a relationship involving the reciprocal of \( x \).

a Plot the graph of \( y = \frac{1}{x} \) after completing this table of values. This graph is called a hyperbola.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(-\frac{1}{2})</th>
<th>(-\frac{1}{3})</th>
<th>(-\frac{1}{4})</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-\frac{1}{3})</td>
<td>(-2)</td>
<td>(2)</td>
<td>(4)</td>
<td>(6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b You will obtain a graph similar to the sketch graph shown. Can you explain why you don’t join the plotted points at \( x = -\frac{1}{4} \) and \( x = \frac{1}{4} \)? (Hint: what is the \( y \) value when \( x = 0 \)?)

c An important feature of the graph of \( y = \frac{1}{x} \) is the lines that form boundaries for each part of the graph. These lines or asymptotes lie on the \( x \)- and \( y \)-axes.

Write the rule of the asymptote for \( y = \frac{1}{x} \) that lies on the:

i \( x \)-axis
ii \( y \)-axis.

d The sketch graphs of \( y = \frac{1}{x} \) and \( y = \frac{1}{x - 2} \) are shown at right.

i Identify the rule of each asymptote (shown in red) for the graph of \( y = \frac{1}{x - 2} \).

ii Describe how you could use a translation to produce the graph of \( y = \frac{1}{x - 2} \) from the graph of \( y = \frac{1}{x} \). (Hint: consider the position of the asymptotes.)

e Use your understanding of translations to describe how the graphs of these relationships can be produced from the graph of \( y = \frac{1}{x} \).

i \( y = \frac{1}{x} + 2 \)
ii \( y = \frac{1}{x} - 3 \)
iii \( y = \frac{1}{x - 4} \)
iv \( y = \frac{1}{x + 1} \)
v \( y = \frac{1}{x - 3} + 2 \)

f Use your answers to part e to sketch each relationship. Clearly show the asymptotes for each graph.

g Use a calculator or other digital technology to produce the same graphs.

18 Consider the relationships in questions 15–17.

a Sketch the graphs of \( y = -x^3 \), \( y = -x^3 + 4 \) and \( y = -(x - 3)^3 \) on the same Cartesian plane. (Hint: first reflect the graph of \( y = x^3 \) in the \( x \)-axis.)

b Sketch the graphs of \( y = -\sqrt{x} \), \( y = -\sqrt{x} + 3 \) and \( y = -\sqrt{x} - 4 \) on the same Cartesian plane. (Hint: first reflect the graph of \( y = \sqrt{x} \) in the \( x \)-axis.)

c Sketch the graphs of \( y = -\frac{1}{x} \), \( y = -\frac{1}{x} + 4 \) and \( y = -\frac{1}{x - 5} \) on the same Cartesian plane. (Hint: first reflect the graph of \( \frac{1}{x} \) in the \( x \)-axis.)

d Use a calculator or other digital technology to produce the same graphs.

Reflect

How can you identify the radius and the coordinates of the centre of a circle from its rule?
**4G Relationships and direct proportion**

**Start thinking!**

In some relationships between two variables, one variable will increase in direct proportion to the other.

1. Consider the table of values on the right.
   - Plot the points on a Cartesian plane and join them with a smooth line.
   - What type of relationship do you see (linear or non-linear)?
   - As \( x \) increases, does \( y \) increase or decrease? Is this change constant? Explain.
   - One way to compare the rate of change for \( x \) and \( y \) is to divide each \( y \) value by its corresponding \( x \) value; that is, work out \( \frac{y}{x} \). Calculate this value for each pair of coordinates except (0, 0). What do you notice?
   - What is the gradient of this graph? What does this tell you about the rate of change for \( x \) and \( y \)?

2. This relationship is an example of direct proportion, as \( y \) increases at a constant rate with respect to \( x \). This means that \( y \) is directly proportional to \( x \), or \( y \propto x \). What features do you see from the graph that show there is direct proportion? (Hint: what type of graph is it and where does the graph cross the axes?)

3. The general form of the rule for this type of graph is \( y = mx \), where \( m \) is the gradient. When working with proportion, this is often written as \( y = kx \), where \( k \) is the constant of proportionality. Use your answer to question 1e to write the rule for the relationship above.

**KEY IDEAS**

- The relationship between \( x \) and \( y \) will show direct proportion if:
  - \( y \) increases as \( x \) increases
  - its graph is a straight line passing through (0, 0)
  - the rate of change (gradient or the value of \( \frac{y}{x} \) for each coordinate pair) is constant.

- If \( y \) is directly proportional to \( x \) (or \( y \propto x \)), the rule for the relationship is \( y = kx \), where \( k \) is the constant of proportionality and \( k = \text{rate of change} = \text{gradient} \).

- If \( y \) is directly proportional to \( x^2 \) or \( x^3 \) or \( \sqrt{x} \) or \( \frac{1}{x} \), the rule will be \( y = kx^2 \) or \( y = kx^3 \) or \( y = k\sqrt{x} \) or \( y = \frac{k}{x} \).
EXERCISE 4G  Relationships and direct proportion

EXAMPLE 4G-1  Identifying whether a relationship shows direct proportion

Identify whether each relationship shows direct proportion between \(x\) and \(y\) by calculating \(\frac{y}{x}\) for each coordinate pair.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>7</td>
<td>14</td>
<td>21</td>
<td>28</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

**THINK**

1. To check whether there is a constant rate of change for \(x\) and \(y\), calculate \(\frac{y}{x}\) for each pair of coordinates except \((0, 0)\).

2. State whether there is direct proportion.
   - A constant rate of change is required with the graph of the relationship forming a straight line through \((0, 0)\).

**WRITE**

\[
\begin{align*}
\frac{y}{x} &= \frac{7}{1} = 7; \quad \frac{14}{2} = 7; \\
\frac{21}{3} &= 7; \quad \frac{28}{4} = 7
\end{align*}
\]

As there is a constant rate of change starting from \((0, 0)\), there is direct proportion between \(x\) and \(y\).

\[
\begin{align*}
\frac{y}{x} &= \frac{5}{3} = 3; \quad \frac{27}{9} = 3; \quad \frac{4}{4} = 1
\end{align*}
\]

There is no constant rate of change so there is no direct proportion between \(x\) and \(y\).

1 Identify whether each relationship shows direct proportion between \(x\) and \(y\) by calculating \(\frac{y}{x}\) for each coordinate pair.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>27</td>
<td>64</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c)</th>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>0</td>
<td>9</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(d)</th>
<th>(x)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

2 For each relationship in question 1 that shows direct proportion between \(x\) and \(y\):  
   - i plot its graph  
   - ii find the gradient of the linear graph  
   - iii compare the gradient to the value of \(\frac{y}{x}\) for each coordinate pair  
   - iv write its rule using \(y = kx\), where \(k\) is the gradient of the linear graph.
EXAMPLE 4G-2  Finding the rule for a linear relationship using direct proportion

Find the rule for each relationship.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong></td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
</tr>
<tr>
<td><strong>b</strong></td>
<td>0</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

**THINK**

1. Calculate $\frac{y}{x}$ for each pair of coordinates except (0, 0). (Alternatively, plot the graph.)

2. State whether there is direct proportion between $x$ and $y$.

3. Write the proportion statement and general form of the rule. Use the constant value of $\frac{y}{x}$ for $k$.

(Alternatively, find the gradient of the linear graph.)

**WRITE**

- For **a**:
  \[
  \frac{y}{x} = \frac{7}{1} = 7; \quad \frac{y}{x} = \frac{14}{2} = 7
  \]
  
  As there is a constant rate of change, $y$ is directly proportional to $x$.
  
  $y \propto x$
  
  $y = kx$ where $k$ is 7.
  
  Rule is $y = 7x$.

**EXAMPLE 4G-2**

Finding the rule for a linear relationship using direct proportion

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>x</strong></td>
<td>0</td>
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<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td><strong>y</strong></td>
<td>0</td>
<td>7</td>
<td>14</td>
<td>21</td>
<td>28</td>
</tr>
</tbody>
</table>

**Understanding and Fluency**

3. Find the rule for each relationship.

4. Consider this relationship.

   a. Plot the points on a Cartesian plane and join them with a smooth line.

   b. What type of relationship do you see? (Linear or non-linear?)

   c. As $x$ increases, does $y$ increase or decrease? Is this change constant? Explain.

   d. Calculate the value of $\frac{y}{x}$ for each pair of coordinates. What do you notice?

   e. Is the relationship between $x$ and $y$ an example of direct proportion? Explain.

   f. Copy and complete this table of values for the relationship. Instead of $x$ values, look at $x^2$ values.

   g. Copy the Cartesian plane shown, then plot the points from the table in part f and join them with a smooth line.

   h. Is the relationship between $x^2$ and $y$ an example of direct proportion? Explain.

   i. Suggest how you could use this graph to write the rule for the relationship. Discuss with a classmate.

   j. Since $y$ is directly proportional to $x^2$, you can write $y \propto x^2$.

   So the rule will be of the form $y = kx^2$, where $k$ is a constant value. Find the gradient of the linear graph and hence write the rule for this relationship.

   k. Check that your rule is correct by substituting a pair of $x$ and $y$ values, such as $x = 2$ and $y = 8$. 
EXAMPLE 4G-3 Finding the rule for a non-linear relationship using direct proportion

Find the rule for this relationship.

**THINK**

1. Check whether there is a constant rate of change for \( x \) and \( y \). Calculate \( \frac{y}{x} \) for each pair of coordinates except (0, 0). (Alternatively, plot the graph.)

2. State whether there is direct proportion between \( x \) and \( y \).

3. Create a table of values for \( x^2 \) and \( y \). Check whether there is a constant rate of change for \( x^2 \) and \( y \). Calculate \( \frac{y}{x^2} \) for each pair of coordinates except (0, 0). (Alternatively, plot the graph.)

4. State whether there is direct proportion between \( x^2 \) and \( y \).

5. Write the proportion statement and general form of the rule. Use the constant value of \( \frac{y}{x^2} \) for \( k \). (Alternatively, find the gradient of the linear graph.)

**WRITE**

\[
\begin{align*}
\frac{y}{x} &= \frac{5}{1} = 5; \quad \frac{y}{x} = \frac{20}{2} = 10 \\
\frac{y}{x} &= \frac{45}{3} = 15; \quad \frac{y}{x} = \frac{80}{4} = 20
\end{align*}
\]

As there is not a constant rate of change, \( y \) is not directly proportional to \( x \). Try \( x^2 \) and \( y \).

\[
\begin{array}{cccc}
\text{x}^2 & 0 & 1 & 4 & 9 & 16 \\
y & 0 & 5 & 20 & 45 & 80
\end{array}
\]

\[
\begin{align*}
\frac{y}{x^2} &= \frac{5}{1} = 5; \quad \frac{y}{x^2} = \frac{20}{4} = 5 \\
\frac{y}{x^2} &= \frac{45}{9} = 5; \quad \frac{y}{x^2} = \frac{80}{16} = 5
\end{align*}
\]

As there is a constant rate of change starting from (0, 0), \( y \) is directly proportional to \( x^2 \).

\( y \propto x^2 \)

\( y = kx^2 \) where \( k \) is 5.

Rule is \( y = 5x^2 \).

---

**5** Find the rule for each relationship.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
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<td>48</td>
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</tbody>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>7</td>
<td>28</td>
<td>63</td>
<td>112</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>16</td>
<td>64</td>
<td>144</td>
<td>256</td>
</tr>
</tbody>
</table>

**6** State whether each graph shows direct proportion. Provide a reason for your answer.
7 Write the constant of proportionality for each rule.
   \[ y = 4x^2 \]  \[ h = 3.5t \]  \[ b = 6\sqrt{a} \]  \[ m = 10n^3 \]  \[ y = \frac{2}{x} \]

**EXAMPLE 4G-4** Finding the constant of proportionality from given information

Find \( k \), the constant of proportionality, using the given information in each case.

a \( y = kx \) and \( y = 18 \) when \( x = 3 \)

b \( y = kx^2 \) and \( y = 36 \) when \( x = 2 \)

**THINK**

a Substitute the known values for \( x \) and \( y \) into the rule and solve for \( k \).

b Substitute the known values for \( x \) and \( y \) into the rule and solve for \( k \).

**WRITE**

a \( y = kx \)
   
   When \( x = 3 \), \( y = 18 \) so \( 18 = k \times 3 \)
   
   \( 18 = 3k \)
   
   \( k = 6 \)

b \( y = kx^2 \)
   
   When \( x = 2 \), \( y = 36 \) so \( 36 = k \times 2^2 \)
   
   \( 36 = 4k \)
   
   \( k = 9 \)

8 Find \( k \), the constant of proportionality, using the given information in each case.

a \( y = kx \) and \( y = 50 \) when \( x = 5 \)

b \( y = kx^2 \) and \( y = 72 \) when \( x = 3 \)

c \( y = kx^3 \) and \( y = 32 \) when \( x = 2 \)

d \( y = k\sqrt{x} \) and \( y = 21 \) when \( x = 9 \)

e \( y = \frac{k}{x} \) and \( y = 11 \) when \( x = 4 \)

f \( y = kx^2 \) and \( y = 18 \) when \( x = 6 \)

9 Find the constant of proportionality using the given information in each case.

a \( y \propto x \) and \( y = 20 \) when \( x = 4 \)

b \( p \propto q^2 \) and \( p = 20 \) when \( q = 4 \)

c \( d \propto e^3 \) and \( d = 250 \) when \( e = 5 \)

d \( h \propto \sqrt{g} \) and \( h = 70 \) when \( g = 100 \)

e \( a \propto \frac{1}{m} \) and \( a = 9 \) when \( m = 3 \)

f \( w \propto v \) and \( w = 15 \) when \( v = 6 \)

10 Write the rule for each relationship in question 9.

11 Find the constant of proportionality in each case. (Hint: find the gradient of each straight line.)

a \( y \propto x^2 \)

b \( y \propto \sqrt{x} \)

C \( y \propto \frac{1}{x} \)

12 Write the rule for each relationship in question 11.
13 Julia listed the cost in dollars ($c$) for different numbers of bread rolls ($n$) in a table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
</tbody>
</table>

a. Plot the points on a Cartesian plane and join them with a smooth line.
b. What type of relationship do you see?
c. Is the relationship between $n$ and $c$ an example of direct proportion? Explain.
d. Write the proportion statement and general form of the rule.
e. Find the gradient of the linear graph and hence write the rule for this relationship.
f. What is the cost of 20 bread rolls?

14 Tom is riding in a cycling event. His distance from the start line at given times is recorded. The table shows values for $t$ (number of hours) and $d$ (distance from the start line in km).

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>4</th>
<th>9</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>0</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sqrt{t}$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>0</td>
<td>20</td>
<td>40</td>
<td>60</td>
</tr>
</tbody>
</table>

a. Plot the points on a Cartesian plane and join them with a smooth line.
b. What type of relationship do you see?
c. Is the relationship between $t$ and $d$ an example of direct proportion? Explain.
d. Copy and complete this table.
e. Plot the points on a Cartesian plane and join them with a smooth line.
f. Is the relationship between $\sqrt{t}$ and $d$ an example of direct proportion? Explain.
g. Write the proportion statement and general form of the rule.
h. Find the gradient of the linear graph and hence write the rule for this relationship.
i. What distance is Tom from the start line after 36 hours?

15 A bus is hired for students to attend a theatre night. The cost to each student depends on how many students agree to go. Some examples are shown in the table where the cost per student in dollars ($c$) is provided for different numbers of students ($n$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>25</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\frac{1}{n}$</th>
<th>1</th>
<th>$\frac{1}{2}$</th>
<th>$\frac{1}{4}$</th>
<th>$\frac{1}{8}$</th>
<th>$\frac{1}{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>200</td>
<td>100</td>
<td>50</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

a. Plot the points on a Cartesian plane and join them with a smooth line.
b. What type of relationship do you see?
c. Is the relationship between $n$ and $c$ an example of direct proportion? Explain.
d. Copy and complete this table.
e. Plot the points on a Cartesian plane and join them with a smooth line.
f. Is the relationship between $\frac{1}{n}$ and $c$ an example of direct proportion? Explain.
g. Write the proportion statement and general form of the rule.
h. Find the gradient of the linear graph and hence write the rule for this relationship.
i. What is the cost of hiring the bus?
j. What is the cost per student if 12 students wish to go on the bus?

**Reflect**

How can direct proportion be used to work out the rule for a non-linear relationship?
CHAPTER REVIEW

SUMMARISE

Create a summary of this chapter using the key terms below. You may like to write a paragraph, create a concept map or use technology to present your work.

- quadratic equation
- Null Factor Law
- linear relationship
- non-linear relationship
- parabola
- sketch
- turning point
- minimum turning point
- maximum turning point
- symmetrical
- axis of symmetry
- y-intercept
- x-intercept
- transformation
- dilation
- reflection
- horizontal translation
- vertical translation
- upright parabola
- inverted parabola
- radius of circle
- centre of circle
- point of inflection
- hyperbola
- asymptote
- direct proportion
- rate of change
- constant of proportionality

MULTIPLE-CHOICE

4A  1 Which is a quadratic expression?
A  \( y = x^2 + 2 \)
B  \( x = 34 \)
C  \( x^2 - 4x \)
D  \( x = 12y \)

4A  2 Which is a quadratic equation?
A  \( y = x^2 + 2 \)
B  \( x = 34 \)
C  \( x^2 - 4x \)
D  \( x = 12y \)

4B  3 Which of these is a non-linear relationship?
A  \( y = \frac{1}{3}x \)
B  \( y = \frac{x}{3} - 3 \)
C  \( y = \frac{1}{3}x - 4 \)
D  \( y = x^2 - 4 \)

4C  4 Which rule would produce a graph of \( y = x^2 \) translated 5 units down?
A  \( y = x^2 + 5 \)
B  \( y = 5x^2 \)
C  \( y = x^2 - 5 \)
D  \( y = \frac{1}{5}x^2 \)

4C  5 The graph of \( y = -4x^2 + 1 \) is formed from the graph of \( y = x^2 \). Which statement is not true?
A  \( y = x^2 \) has been dilated.
B  \( y = x^2 \) has been reflected in the x-axis.
C  \( y = x^2 \) has been translated vertically.
D  \( y = x^2 \) has been translated horizontally.

4D  6 The coordinates of the turning point of the graph of \( y = -(x - 4)^2 \) are:
A  \((4, 0)\)
B  \((-4, 0)\)
C  \((0, 4)\)
D  \((0, -4)\)

4E  7 The coordinates of the x-intercepts of the graph of \( y = x^2 - 4x - 12 \) are:
A  \((0, -6), (0, 2)\)
B  \((0, 6), (0, -2)\)
C  \((-6, 0), (2, 0)\)
D  \((6, 0), (-2, 0)\)

4E  8 The y-intercept of the graph of \( y = (x - 5)(x + 2) \) is:
A  \(-2\)
B  \(-10\)
C  \(5\)
D  \(10\)

Questions 9 and 10 refer to the graph of \((x - 2)^2 + (y + 4)^2 = 9\).

4F  9 The radius of the circle is:
A  2 units
B  3 units
C  4 units
D  9 units

4F  10 The centre of the circle is located at:
A  \((2, -4)\)
B  \((-2, 4)\)
C  \((4, -2)\)
D  \((-4, 2)\)

4G  11 If \( y \) is directly proportional to \( x \), the graph of the relationship is a:
A  parabola
B  circle
C  straight line
D  hyperbola
SHORT ANSWER

4A  1 Find the solution/s to each equation. For any that do not have a solution, provide a reason.
   a \( x^2 - 5x + 6 = 0 \)  b \( x^2 + x - 30 = 0 \)
   c \( x^2 + 9 = 0 \)  d \( x^2 - 12x = 0 \)

4B  2 Plot each relationship and use your graph to identify:
   i whether the parabola is upright or inverted
   ii the type of turning point and its coordinates
   iii the x- and y-intercepts.
   a \( y = 4x^2 - 4 \)  b \( y = -4x^2 - 4x \)
   c \( y = x^2 - 4x + 4 \)

Questions 3 and 4 refer to these quadratic relationships.
   a \( y = 5x^2 \)  b \( y = \frac{1}{3}x^2 \)
   c \( y = -5x^2 \)  d \( y = x^2 + 5 \)
   e \( y = -x^2 - 5 \)  f \( y = -\frac{1}{3}x^2 \)

4C  3 Classify each graph as:
   i upright or inverted
   ii wider or narrower than the graph of \( y = x^2 \)
   iii having a minimum or maximum turning point.

4D  4 Describe the transformation/s needed to produce each graph from \( y = x^2 \).

4D  5 Describe the parabola produced by each rule. Identify whether the curve is upright or inverted, and list the transformations that have been performed on the graph of \( y = x^2 \).
   a \( y = -5(x + 2)^2 - 1 \)  b \( y = 5x^2 + 4 \)
   c \( y = \frac{1}{4}(x - 5)^2 \)  d \( y = -3x^2 \)

4D  6 Sketch the graph of each quadratic relationship, clearly showing the coordinates of the turning point.
   a \( y = (x + 1)^2 - 1 \)  b \( y = -(x + 3)^2 - 3 \)
   c \( y = (x - 4)^2 + 4 \)  d \( y = -(x - 2)^2 - 2 \)

4E  7 Find the coordinates of the x- and y-intercepts for the graph of each rule.
   a \( y = -x^2 - 4x \)  b \( y = x^2 - x - 12 \)
   c \( y = (x + 5)(x - 4) \)  d \( y = -(x + 2)(x + 1) \)

4E  8 Sketch each graph described in question 7, showing the turning point.

4F  9 Sketch each circle on a Cartesian plane. Clearly identify the centre and radius.
   a \( (x + 4)^2 + y^2 = 9 \)  b \( x^2 + (y + 3)^2 = 16 \)
   c \( (x - 4)^2 + (y - 3)^2 = 16 \)  d \( (x + 3)^2 + (y + 5)^2 = 49 \)

4G  10 Use the graph of \( y = x^3 \), \( y = \sqrt{x} \) or \( y = 2x \) to describe and sketch the graph of each relationship.
   a \( y = x^3 + 4 \)  b \( y = (x + 4)^3 \)
   c \( y = \sqrt{x} + 3 \)  d \( y = \sqrt{x} + 3 \)
   e \( y = \frac{1}{x} - 2 \)  f \( y = \frac{1}{x - 2} \)

4H  11 Find the value of the constant of proportionality for each direct proportion relationship.
   a \( y \propto x^2 \) and \( y = 4 \) when \( x = 2 \)
   b \( y \propto x^3 \) and \( y = 2 \) when \( x = \frac{1}{2} \)
   c \( y \propto \sqrt{x} \) and \( y = 2 \) when \( x = 16 \)
   d \( y \propto \frac{1}{x} \) and \( y = 8 \) when \( x = \frac{1}{2} \)

4I  12 A direct proportion relationship exists in each of these. Find each rule.
   a
   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
   y & 0 & 5 & 10 & 15 & 20 \\
   \end{array}
   \]
   b
   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
   y & 0 & 0.5 & 2 & 4.5 & 8 \\
   \end{array}
   \]
   c
   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
   y & 0 & 3 & 24 & 81 & 192 \\
   \end{array}
   \]
   d
   \[
   \begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
   y & 0 & 3 & 6 & 15 & 18 \\
   \end{array}
   \]
NAPLAN-STYLE PRACTICE

1. If \((x - 6)(x + 1) = 0\), the two solutions for \(x\) are:
   - \(x = -6\) or \(x = 1\)
   - \(x = 6\) or \(x = -1\)

2. The expanded form of the equation \((x - 8)(x + 2) = 0\) is:
   - \(x^2 + 6x - 16 = 0\)
   - \(x^2 - 6x - 16 = 0\)
   - \(x^2 - 10x - 16 = 0\)
   - \(x^2 + 10x - 16 = 0\)

3. What is the solution to \(x^2 - 11x + 10 = 0\)?

4. For \(y = x^2 + 3x - 2\), what is the value of \(y\) when \(x = -1\)?

5. Write the coordinates of the \(x\)-intercepts for the graph of \(y = x^2 + x - 20\).

6. The axis of symmetry for the graph of \(y = x^2 + 2x - 15\) has the rule:
   - \(x = 1\)
   - \(x = -1\)
   - \(y = 1\)
   - \(y = -1\)

7. Write the coordinates of the turning point for the graph of \(y = x^2 - 8x + 15\).

8. What transformation has been performed on the graph of \(y = x^2\) to produce the graph of \(y = x^2 - 3\)?
   - dilation by factor of 3
   - horizontal translation of 3 units
   - reflection in the \(x\)-axis
   - vertical translation of 3 units

9. Which relationship would produce the narrowest parabola when graphed on the same Cartesian plane?
   - \(y = x^2 + 4\)
   - \(y = -4x^2 - 5\)
   - \(y = \frac{1}{2}x^2 + 1\)
   - \(y = 2x^2 + 3\)

10. What are the coordinates of the \(y\)-intercept for the graph of \(y = -3(x - 2)^2 - 4\)?

11. What are the coordinates of the turning point for the graph of \(y = -3(x - 2)^2 - 4\)?

12. Which rule best matches the graph shown?
   - \(y = (x - 3)^2 + 4\)
   - \(y = (x - 3)^2 - 4\)
   - \(y = (x + 3)^2 + 4\)
   - \(y = (x - 3)^2 + 4\)

13. How many \(x\)-intercepts does the graph of \(y = (x - 3)^2\) have?
   - 0
   - 1
   - 2
   - 3

14. Which rule best matches the graph shown?
   - \(y = x^2 + 2x + 3\)
   - \(y = -x^2 + 2x + 3\)
   - \(y = x^2 - 2x - 3\)
   - \(y = -x^2 + 2x - 3\)

15. A circle has its centre at \((2, -3)\) and a radius of 5 units. What is its rule?
   - \((x - 2)^2 + (y + 5)^2 = 25\)
   - \((x + 2)^2 + (y - 3)^2 = 25\)
   - \((x - 2)^2 + (y - 3)^2 = 5\)
   - \((x + 2)^2 + (y + 3)^2 = 5\)

16. What are the coordinates of the centre and the radius of the circle with the rule \(x^2 + (y - 8)^2 = 4^2\)?
   - \((0, -8), 2\) units
   - \((0, 8), 2\) units
   - \((0, 8), 4\) units
   - \((8, 0), 4\) units

17. What is the area of the circle with the rule \((x - 2)^2 + (y + 5)^2 = 36\)? Write your answer to the nearest square unit.

18. Which rule best matches the graph shown?
   - \(y = x^2\)
   - \(y = \frac{1}{x}\)
   - \(y = x^3\)
   - \(y = \sqrt{x}\)
1 If a vertical cut is made through the centre of this bowl, the inside edge of the cross-section has the shape of a parabola. This cross-section can be modelled by the quadratic relationship \( y = (x - 4)^2 + 1 \), where \( y \) is the height of the cross-section in centimetres above the table at a horizontal distance of \( x \) cm from the left side of the cross-section.

a On a Cartesian plane, sketch the graph of the quadratic relationship.

b What are the coordinates of the two points representing the top rim of the bowl?

c Determine the diameter of the upper rim of the bowl.

d What is the thickness of the bowl where it is sitting on the table?

2 The outline of a bridge has an upper arch and a lower arch that can each be modelled by a quadratic relationship. If \( h \) is the height of the arch at a horizontal distance \( d \) from the left side of the bridge, the two rules are:

- lower arch: \( h = -\frac{1}{10}d^2 + 4d \)
- upper arch: \( h = -\frac{1}{8}(d - 20)^2 + 65 \).

All measurements are in metres.

a Draw a sketch of the two arches on the same Cartesian plane.

b Comment on the shape and features of the two parabolas.

c What is the height of the upper arch above the lower arch at the ends of the bridge?

d What is the span of the bridge? Show this:

i graphically, by referring to your graphs in part a

ii algebraically, by solving a quadratic equation.

The distance between the arches does not remain constant over the span of the bridge.

e Find the height of the highest point of the:

i lower arch

ii upper arch.

f How far apart are the two arches at their highest point?

g How far apart are the two arches at a horizontal distance of 10 m from the left side of the bridge?

h Describe the distance between the two arches over the span of the bridge.
CHAPTER 4: NON-LINEAR RELATIONSHIPS

CONNECT

Path of a soccer ball

To analyse the path of the ball during a soccer match, Lisa records some short pieces of video so measurements can be taken. The measurements (in metres) indicate the height of the ball \( h \) for a given horizontal distance \( d \) that the ball travels.

She chooses two plays where the ball follows a parabolic path after contact with a player. One is when the ball was kicked by David and the other when Nick hit the ball with his head. Each path can be modelled by a quadratic relationship involving \( d \) and \( h \).

David: \( h = -\frac{1}{100}d(d - 44) \)

Nick: \( h = -\frac{1}{20}(d - 8)^2 + 5 \)

Your task

You are to analyse each relationship to determine:
- the height of the ball when the player made contact with it
- the maximum height of the ball during the play
- the horizontal distance from where the player made contact to where the ball hits the ground (assuming no other player gets to it first).

Include all necessary graphs and working to justify your answers. Where appropriate, show how you can obtain your answers using both graphical and algebraic methods.

Compare your observations for both relationships to determine who kicked/hit the ball highest and furthest. Include an analysis of how far the ball travels if it is intercepted by another player when the ball is 1 m off the ground.

During training, each player dribbles the ball along the ground for the same distance from a marked point to the goal line. Lisa records measurements for the average speed of the ball in metres per second \( s \) and the time in seconds \( t \) for the ball to travel this distance (see table at right).

Use appropriate graphs and calculations to determine the rule for the relationship between \( t \) and \( s \). Use this rule to work out the average speed of the ball if it takes 5 seconds to cover the distance. What distance has the ball travelled in each case?

<table>
<thead>
<tr>
<th>( t )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>24</td>
<td>16</td>
<td>12</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>
As an extension, you may like to create a rule of your own that would describe the path of a soccer ball. Fully explain how you obtained the relationship.

You may like to present your findings as a report. Your report could be in the form of:

- a poster
- a PowerPoint presentation
- a technology demonstration
- other (check with your teacher).