Chapter 1

Understanding school mathematics

Big Ideas

- Sophisticated mathematical skills are inherent in many daily activities.
- School mathematics needs to change and is changing.
- Learning mathematics is both an individual and social enterprise.
- Teaching mathematics is a rewarding but complex and demanding task.

Chapter objectives

This chapter will enable the reader to:

– Understand the contribution of mathematics to society and the role of school mathematics in a futures-oriented curriculum
– Develop a broad understanding of what is involved in the teaching and learning of mathematics in contemporary Foundation to Year 9 classrooms
– Appreciate the range of issues and challenges impacting on the provision of school mathematics at this level
– Recognise the important role of reflection and research in mathematics education.

Key terms

Communication patterns
Conceptual understanding
Curriculum
Knowledge for teaching mathematics
Mathematical problem solving
Mathematical reasoning
Open-ended questions
Pedagogical content knowledge
Pedagogy
Problem solving
Procedural fluency
Representations
Rich tasks

Everyone can do maths

'I was never any good at maths ... I dropped it as soon as I could.' Most teachers of mathematics experience this reaction in a social context when they are asked their occupation. Such responses are amazing when you consider that the same people would probably not have made the same claims about their capacity to read or write. It seems that it is socially acceptable to admit to disliking or not being 'very good at' mathematics. It is a sad irony that those who profess such views frequently demonstrate sophisticated uses of mathematics in their everyday activities. For instance, I recall a taxi driver who, having confessed that he had 'failed mathematics in Year 9', expertly gauged the flow of the traffic, consulted a global positioning system, decided to take an alternative route to ensure we arrived in time, and at the end of the journey mentally added the airport tax to the fare and calculated the change.

1 When was the last time you truly, madly, deeply, really enjoyed doing some mathematics? What did you do and how did it make you feel?
2 Can you recall a teacher who made a significant impact on your learning of mathematics or an experience when you were 'turned off' mathematics? If so, describe the circumstances in terms of who, what, when and where. How did it make you feel?
3 What lessons can be learnt from this?

Introduction

We introduced this book by talking about teacher quality and what it means to be an effective teacher of mathematics. While it is relatively easy to list generic characteristics, what matters is what students experience, individually and collectively. Teacher quality is clearly related to teacher knowledge and confidence, but what actually happens in classrooms is also affected by what teachers feel they have to teach, how they go about teaching it, and the social contexts in which teaching takes place.

Teachers choose to teach for a variety of reasons—and it is rarely for the money! The most commonly cited reason for choosing to teach is 'to make a difference'. In *A Sense of Calling: Who teaches and why*, Farkas, Johnson and Foleno (2000) report that 96 per cent of new teachers surveyed reported that they chose teaching because it 'involves work they love doing' and gave them 'a sense of contributing to society and helping others' (p. 11). 'Understanding what matters to people, what motivates them and why they do what they do can make the difference between a conversation that moves forward and one that goes nowhere' (p. 8).

Given that most teachers are motivated by a desire to support the learning and well-being of others, and given what we know about effective mathematics teaching, why is it that students' experiences of learning mathematics are not that much different from what they were in the past, particularly in the middle years of schooling (Bodin & Capponi, 1996; Yates, 2005)? It appears that teachers tend to teach mathematics in the way that they were taught (e.g. Brady, 2007; Stigler & Hiebert, 1999) and that their decisions about content and pedagogy are very much related to what they know and believe about the nature of mathematics and the teaching and learning of mathematics (e.g. Brady, 2007; Fennema, Carpenter & Peterson, 1989; Handal & Herrington, 2003, Siemon, 1989).

Consider and discuss your teaching

JOB, VOCATION OR CAREER?

For me, in Year 12, teaching was the obvious choice. Where else do you get an opportunity to do all the things you like doing, such as learning mathematics, studying history, discussing literature, playing sport, singing and performing. But most of all, I became a teacher because I somehow felt it was the right thing to do. Helping others learn is not only very satisfying, it increases people's opportunities in life. Education is the key to making the world a better, fairer, more environmentally responsible place.

1 What were your reasons for choosing to become a teacher? Was it because of particular strengths and interests, role models, or opportunities?
2 What do you hope to achieve as a teacher of mathematics?
3 What do you see yourself doing in ten years' time?

In what follows we will look at what we mean by mathematics, the goals of contemporary school mathematics, and the influences, including teacher knowledge and beliefs, that act to promote (or constrain) the teaching and learning of mathematics in the primary and middle years of schooling.

What is mathematics?

We all have views on what constitutes mathematics, and these views shape our decisions about how we teach and learn mathematics. For instance, if mathematics is viewed as a set of universal truths, teachers are more likely to see their task as transferring a given set of facts and skills to students and to view student learning as the capacity to reproduce these facts and skills as instructed. If, on the other hand, mathematics is viewed as a socio-cultural practice, a product of reflective human activity, then it is more likely that teachers will see their task as engaging students in meaningful mathematical practices and view student learning in terms of conceptual change.

Many views have been advanced to describe the nature of mathematics:

> Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. (Courant & Robbins, 1941, p. xv)

> Mathematics reveals hidden patterns that help us understand the world around us … Mathematics is a science of pattern and order. (Mathematical Sciences Education Board, 1989, p. 31)

> [A]ll cultural traditions … find connecting patterns and apparent symmetries throughout nature. Individually and collectively, we accept these patterns as part of the background of our lives: beyond question or doubt, all Australians agree we live in an orderly, knowable universe. Yet the patterns which organise, and the laws which govern, European knowledge and perception apparently have little in common with the patterns which make sense of the Aboriginal world … both kinds of patterning form complex and mathematically sophisticated systems, both have powerful ideological underpinnings, both are social constructions emerging from historically identifiable contexts, both attempt to account for natural as well as social phenomena, both are rational in principle and practical in application. (Watson, 1989, p. 31)

These views all propose that mathematics offers a way of understanding the world we live in. Mathematics provides a consistent framework, a symbolic technology, by which we can model 'reality', solve problems, and support predictions, but how this is described and communicated depends very much on our cultural values and traditions. For instance, our taken-for-granted view of mathematics—the mathematics of our schooling that Bishop (1991) refers to as 'Mathematics with a capital "M"'—is governed by number patterns and relations. There are other cultures whose ways of understanding the world are not governed by 'how much and how many' but by complex kinship patterns that connect all things to each other without the need of numbers (e.g. Watson, 1989; Bishop, 1991). In other words, mathematics is a 'pan-cultural phenomenon: that is, it exists in all cultures', and 'Mathematics' is a 'particular variant of mathematics, developed through the ages by various societies' (Bishop, 1991, p. 19).

This is a difficult notion to grapple with, as we can scarcely imagine a world without Mathematics, and yet we happily accept that different languages are developed by different cultural groups in order to communicate. This prompted Bishop to ask: if language develops from the need for, and activity of, *communicating*, what are 'the activities and processes which lead to the development of mathematics?' (1991, p. 22). According to Bishop, there are six fundamental mathematical activities that occur in some form across all cultures, which he describes as follows.

- *Counting*—the use of a systematic way to compare and order discrete phenomena. It may involve tallying, using objects or string to record, or special number words or names
- *Locating*—exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means
- *Measuring*—quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'
- *Designing*—creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object as a 'mental template', or symbolising it in some conventional way
- *Playing*—devising, and engaging in, games and pastimes, with more or less formalised rules of play that all players abide by
- *Explaining*—finding ways to account for the existence of phenomena, be they religious, animistic or scientific (Bishop, 1988, pp. 182–3).

In looking at these activities and processes, we can see how it might be possible for there to be many mathematics underpinned by different cultural norms and values. For Bishop, mathematics can be viewed as a 'way of knowing'. The cultured nature of mathematics is illustrated by an apocryphal story that describes a group of tourists who were being escorted through the central Australian desert by a local Indigenous man. When they had been travelling for some time and were feeling overwhelmed by the seemingly unchangeable landscape, one of the tourists asked the guide, 'What do you do when you get lost out here?' The man turned to the tourist somewhat bemused, and said, 'I go home.' In other words, for him, there was no notion of 'lost'. He knew the landscape intimately, and he could discern its patterns and variations. By this means he knew exactly where he was and in which direction he needed to travel to return home, without the need of GPS, compass or paper map.

Mathematics as we know it evolved from the earliest civilisations of Egypt, Greece, Mesopotamia, India and China. These were established in river valleys where there was plenty of water, fertile soil, and a climate conducive to an agrarian economy. It is not too difficult to imagine a scenario where competition for land and resources to sustain rapidly growing populations led to a system of quantifying to settle disputes and support trade. Land could be identified by precise measures, permanent structures could be erected, goods could be exchanged on a comparable basis, taxes could be levied on property and produce, time could be measured by reference to the sun and the stars, and journeys could be described in terms of units of length, time and direction. When trade and travel expanded, systems of quantifying were used to transcend language barriers.

Mathematics was also pursued as an intellectual pursuit in its own right. For example, the Pythagoreans noticed that for right-angled triangles, the sum of the squares on the smaller sides was equal to the square on the longest side, the hypotenuse. At the time, the only numbers that were recognised were the natural numbers (1, 2, 3, 4, …) and numbers that were ratios of these numbers (e.g. 1:2, $\frac{4}{2}$, or $\frac{17}{3}$). Subsequent explorations of right-angled triangles in which the two smaller sides were both one unit in length led to the proposition of a new type of number, in this case the square root of 2. Although resisted at first, this proposition eventually led to the recognition of the irrational numbers, which are numbers that cannot be expressed as ratios of natural numbers. The irrational numbers include pi (π) and non-terminating decimals. This example illustrates an interesting philosophical question: were the irrational numbers always there, 'waiting to be discovered', or were they the product of human reflective activity at a certain time in a certain cultural setting? This question is often characterised as a debate between *absolutist* and *fallibilist* views of mathematics—that is, between mathematics as a set of irrefutable truths, and mathematics as a socially constructed practice (e.g. Bishop, 1991; Ernest, 1991; Presmeg, 2007). While these views may co-exist to some extent, what teachers believe about the fundamental nature of mathematics has important implications for practice as we shall see below.

Goals of school mathematics

Niss (1996) suggests that there are three fundamental reasons for the existence of school mathematics as we know it today. They are that the study of mathematics contributes to:

> the technological and socio-economic development of society; the political, ideological and cultural maintenance and development of society; [and] the provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens. (p. 22)

In ancient times, the study of mathematics was restricted to a select few, generally young men with the time to converse with more learned others. For some, this was simply an enjoyable pastime, an intellectual pursuit for its own sake, while for others such as the Pythagoreans, in Greece, this activity was shrouded in secrecy and mysticism. However, as production became more specialised and trade expanded, mathematical know-how was acquired as a matter of necessity. Merchants, builders, navigators, tax collectors, artisans and religious leaders all needed some mathematics, and they inducted those that followed them into their particular mathematical practices.

Before the nineteenth century, there was little offered in the way of formal education beyond the opportunity to learn a trade. For the very few who did receive some sort of formal education, this was largely justified on the grounds that it contributed to 'the political, ideological and cultural maintenance of society' (Niss, 1996, p. 22). This situation changed with the advent of the industrial age and the introduction of compulsory schooling. Elementary mathematics was included as a core component of the curriculum, presumably 'to contribute to the technological and socio-economic development of society, while at the same time placing some emphasis on equipping individuals with tools for mastering their vocational and everyday private lives' (Niss, 1996, p. 23).

Consider and discuss your maths

FROM AN 1875 BOOK OF ARITHMETICAL EXAMPLES

> A wine merchant buys 3 hhds. of wine at Bordeaux, at £15 per hhd., pays duty 1 s. per gallon, and carriage £3. What must he sell the wine at per gallon, to clear £10 in the transaction? (Davis, 1875, p. 47)

1 What does this say about the assumptions underpinning the teaching and learning of mathematics at the time? How different are these from the assumptions underpinning your experience of school mathematics? Discuss in terms of the three fundamental reasons for studying mathematics described by Niss above.
2 What knowledge and skills are needed to solve this problem? (Hint: you may need to rewrite this in a more familiar form first.)

In the same book, a 'simple multiplication' problem was given as 856 439 082 × 7 008 001 (p. 13). These days, it is difficult to understand why this was regarded as 'simple' or why such a problem would be set but these examples nicely illustrate that school mathematics is a human construction reflecting societal values and priorities at particular points in time. The fact that all students, not just a select few, were expected to solve these problems is indicative of the goals of school mathematics at the time and the assumptions made about the teaching and learning of mathematics. Contrast this with how an Aboriginal child living in a remote part of Australia in 1875 might have been expected to learn to find his or her way home in the desert.

A shift in emphasis

Today, all three of Niss's reasons for including mathematics in school curricula can be found in contemporary curriculum documents, for example, the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the

Australian Mathematics Curriculum (Australian Curriculum Assessment & Reporting Authority [ACARA], 2015), which aims to ensure that students:

> are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens ... develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, able to pose and solve problems and reason in number and algebra; measurement and geometry; and statistics and probability ... [and] recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study. (p. 1)

Think + link: The chapters in Part 3 examine what we mean by numeracy and explore its relationship to school mathematics in Years F–9. In particular, these chapters focus on the 'big ideas' in mathematics and how these can be applied in everyday contexts to solve problems and inform decision making.

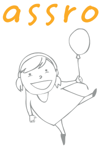
It is interesting to note the order in which these aims are stated. The first echoes Niss's (1996) third reason for studying mathematics, that is, the 'provision of individuals with prerequisites which may assist them in coping with private and social life, whether in education, occupation, or as citizens' (p. 22). This reason for studying mathematics has become much more prominent recently because of its association with *numeracy*, which is defined as the 'effective use of mathematics to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life' (National Numeracy Benchmarking Taskforce, 1997, p. 30).

This shift in emphasis arose in response to concerns about the capacity of individuals to function effectively in a modern technological society, in particular to critically question public policy decisions and understand the ways in which mathematics is being used to monitor and shape our lives (Gellert & Jablonka, 2008).

We have never needed mathematics more

While it is true that technology has replaced many of the routine procedures traditionally taught in school mathematics, the advent of sophisticated information and communication technologies has also drastically changed the way we conduct our everyday lives. Mathematics underpins much of this technology, and mathematics is used in increasingly powerful and subtle ways to persuade voters and consumers to act in certain ways. This suggests that we have never needed mathematics more. But the mathematics we need is not the school mathematics of the past with its emphasis on isolated skills and rote learning.

For the classroom



ACTIVITY 1.1 **Images of mathematics in use**

Collect a range of newspapers, magazines, food containers and screen dumps from a variety of websites. Ask students to identify where mathematics is being used and how. Encourage younger students to look beyond numbers to measures, and older students to look for instances of proportion and other, more subtle, uses of mathematics, such as location of visual displays and headings. Represent as a collage of annotated images for classroom display.

Today, individuals need to be able to make sense of vast amounts of quantitative and spatial information presented in increasingly sophisticated multimedia formats; make decisions on the basis of that understanding; and communicate their reasons for doing so if challenged. According to Becker and Selter (1996), the 'ultimate objective of student learning at all levels is the acquisition of a *mathematical disposition* rather than the absorption of a set of isolated concepts or skills' (p. 542). In their opinion, students should learn to:

> *be creative*: to look for patterns, make conjectures, generate new problems … to *reason*: to give arguments, uncover contradictions, distinguish between facts and assertions … to *mathematize*: to collect data, process information, interpret data and solutions … and to *communicate*: to express their own thoughts, accept the ideas of others, establish forms of cooperation. (p. 542)

This is consistent with Bishop's view of mathematics as 'a way of knowing' rather than 'a way of doing'. The values they represent are reflected in the *Australian Curriculum: Mathematics* (ACARA, 2015) in the form of proficiencies, specifically:

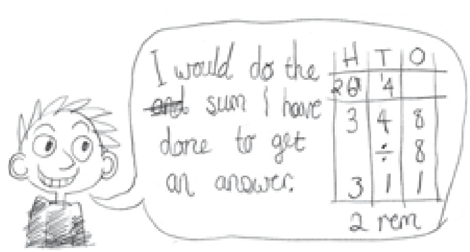
- conceptual understanding
- procedural fluency
- problem solving (or strategic competence), and
- adaptive reasoning.

Think + link: Chapter 4 considers what is involved in building mathematical proficiency and the confidence and competence to view the world mathematically.

From the student's perspective

Student responses to mathematical tasks can reveal a lot about their perceptions of the goals of school mathematics. Nick was a Year 4 student who knew his number facts and was 'good at' 'doing division' by the traditional 'goes into' [sometimes referred to as 'guzinta'] algorithm. For example, he could confidently divide 364 by 7 by saying '7 goes into 3? No, carry the 3. 7 goes into 36 … yes, 5 times and 1 left over [writes 5 underneath and 1 beside the 4], 7 goes into 14, yes 2 times [writes 2]'. After some time in class exploring an alternative approach involving sharing and base 10 materials, which Nick clearly understood and could justify to his peers, he was given the problem 'Eight families shared a prize of $348. How much did each family receive?' Nick provided the following 'talking head' response. He drew a face to show how he felt about the task and wrote a brief explanation of what he had done.

Figure 1.1 Nick's response to a division problem



Source: Siemon, 1993

Think + link: Chapter 2 examines the role of beliefs and values in learning school mathematics and stresses the importance of conceptual understanding as a basis for procedural fluency.

Despite his demonstrated understanding of both forms of division, Nick chose to create his own algorithm based on what he knew about subtraction and renaming. In this case, reading from the top down and starting with the ones he reasoned, '8 how many 8s? … 1 … 4 how many 8s? Can't do, so trade a hundred, 14 tens how many 8s? … 1 and 6 over', which he records over the 2 in the hundreds place. Realising '2 how many 8s?' is not going to work, he crosses the 6 out and rewrites the 2 and the 6 as 26 and proceeds, '26 how many 8s? … 3 and 2 remainder', which he records. His comments indicate his beliefs about what he believes school mathematics is about, that is, using 'sums' to get answers. When asked about his answer, Nick, said, 'Oh if it was real money I wouldn't do it like that.' Prompted to explain how he would do it, Nick replied, 'Well 8 families, $40 each that'd be $320, $50 each would be $400, I reckon it's about $43.' Nick's problem was not with division, but with the values and beliefs he held about the nature and purpose of school mathematics. Asked why he did this, Nick said that he knew his 'old way of doing it would work but Mrs … didn't like that' and he could do it the new way 'but that was too long'.

Affordances and constraints

Teaching mathematics is a complex, demanding, but rewarding task that requires a knowledge of students, of content, and of how that content might best be represented to engage and support learners. Referred to by Shulman (1986) as **pedagogical content knowledge** (PCK) and, more recently and with respect to mathematics teaching and learning, as **knowledge for teaching mathematics** or KTM (Ball, Thames & Phelps, 2008), it clearly involves much more than a knowledge of mathematics at the level taught.

> Teaching is about building relationships—between students and the teacher and among students themselves around mathematics—and engaging together in constructing mathematical meaning. Teaching involves orchestrating the content, the representation, and the people in relation to one another. It is about making decisions in the moment that serve the individuals and collective. It is about understanding the students … It is about working together to negotiate meaning. (Franke, Kazemi & Battey, 2007, p. 228)

Think + link: The characteristics of effective teachers are discussed in more detail in Chapter 3 and in Parts 4 and 5 of this book

Teachers need to have a deep understanding of what makes particular mathematics content difficult to learn (content knowledge), what representations and instructional strategies are best suited to the needs of individual students (pedagogy), and how to manage the relationships in which the teaching and learning takes place (social context). Although indistinguishable in practice, these three key aspects of teacher knowledge are briefly described below.

Knowledge of content

In Australia, decisions about which mathematics to teach are largely guided by the *Australian Curriculum: Mathematics* (ACARA, 2015) and supplementary support material developed by state and territory education authorities. The Curriculum is described by Year level (F–12) across three strands and four proficiencies, as shown in Table 1.1.

Table 1.1 Structure of the *Australian Curriculum: Mathematics*

CONTENT STRAND	PROFICIENCIES
Number and Algebra	[Conceptual] Understanding
Measurement and Geometry	[Procedural] Fluency
Statistics and Probability	**Mathematical Problem Solving**
	Mathematical Reasoning

Source: ACARA, 2015

A major shift in the curriculum is the recognition of the importance of *big ideas*. A 'big idea' provides an organising framework that encompasses and connects a number of related ideas and strategies and supports further learning and generalisations. For example, *multiplicative thinking* not only encompasses the various meanings and representations of multiplication and division, together with a range of appropriate solution strategies, but also supports connections between these operations and the base 10 system of numeration, the rational numbers, and generalisations associated with proportional reasoning.

Modern curriculum also attempts to build on what is known about the progression of children's thinking towards the big ideas. Referred to as *learning trajectories* (Simon, 1995), teachers need to be aware of the developmental pathways, intermediate goals, and the instructional strategies, tasks, and representations by which student thinking is likely to be further enhanced.

Think + link: The 'Consider and discuss' and 'Teaching challenges' sections throughout this book are designed to assist you to build your mathematical knowledge for teaching.

The language of mathematics

Building a vocabulary

Teachers need a vocabulary to reflect on their practice, share their thinking and decision making, and engage in further professional learning. In the chapters that follow we will endeavour to introduce the terms commonly used in relation to the teaching and learning of mathematics in a clear and consistent manner. A glossary of key terms is provided at the end of this book.

The many faces of curriculum

Curriculum comes in many different forms. The curriculum produced by education systems is often referred to as the intended curriculum. It sets the standard for what is valued and what will be assessed at a system or national level. However, schools and teachers need to interpret the intended curriculum in the light of their own knowledge and experience and what they know about their particular student population. This can lead to subtle and not-so-subtle variations in the curriculum that are actually translated into practice and assessed. These versions of the curriculum are often referred to as the implemented curriculum and the evaluated curriculum respectively.